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Non finite basing of one number system with constant [1]

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Abstract. This work gives a negative answer to the problem of finding finite base of equations for the real numbers with distinguished constant 1, ordinary product and unary operation $1 - x$.

1 Introduction

In this paper we study the algebra $A_1 = \langle \mathbb{R},\ 1,\ \neg,\ \cdot,\ = \rangle$, where the set $\mathbb{R}$ is a set of real numbers, 1 is distinguished constant, $\neg$ (*negation*) is the unary operation defined as $\neg x = 1 - x$, $\cdot$ is ordinary product of real numbers. Note that these operations are fundamental in Zadeh's fuzzy logic [3], which for the past 45 years has become one of the most rapidly developing areas of mathematics.

In this work we use the standard algebraic notions of algebras and terms [2].

If two terms $t(x_1, x_2, ..., x_n)$ and $\tau(x_1, x_2, ..., x_n)$ coincide syntactically we write

$$t(x_1, x_2, ..., x_n) \equiv \tau(x_1, x_2, ..., x_n).$$

Equation is a formula of the form $t = \tau$ where t and τ are arbitrary terms. When an equation is satisfied in algebra for every value of the variables, we say that the equation is *valid in algebra* and terms t and τ are *equal in algebra*.

When studying any algebra one of the first questions turns out to be the problem of the identification of all equations valid in it.

Definition 1. *Given a set E of valid equations in a given language, a subset E_0 of E is said to be a* ***base*** *for E if every equation in E can be derived from E_0 by the identity axioms and by logical rules.*

By derivation we mean the well known procedure (theorem of Birkhoff [1] of the completeness of the calculus of equationally) whose description is following:

Let $\{b_i(x_1, x_2, ..., x_{n_i}) = \beta_i(x_1, x_2, ..., x_{n_i}) : i \in I\}$ be a base. Then for any valid equation $t = \tau$ in algebra it is possible to build a chain of equal terms $t \equiv t_0 = t_1 = ... = t_k \equiv \tau$, such that each following term is obtained from previous one by changing in it some subterm $b_i(\theta_1, \theta_2, ..., \theta_{n_i})$ to the subterm $\beta_i(\theta_1, \theta_2, ..., \theta_{n_i})$ (or the contrary: changing a subterm $\beta_i(\theta_1, \theta_2, ..., \theta_{n_i})$ to the subterm $b_i(\theta_1, \theta_2, ..., \theta_{n_i})$).

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The finite base problem for a given algebra is the following: is there a finite base for the set of equations which are valid in the algebra?

Earlier [4] the author found the infinite system of equal terms in the algebra $A = \langle \mathbb{R},\ \neg,\ \cdot,\ =\rangle$, which cannot be derived from any finite system of valid equations.

In the present work a negative answer to the finite base problem is given for the set of all valid equations in the algebra A_1.

Since some statements are proved in [4] but needed to understand the following text we give them without prove.

2 Corresponding polynomials, trivially equal and trivial terms

To each term in algebra A (or A_1) corresponds a polynomial, which takes the same values as the term when real numbers are substituted to the variables. The form of the corresponding polynomials is defined by induction on the complexity of terms:

Definition 2. *1) To terms $x, y, ...(, 1)$ of complexity 0* ***correspond the polynomials*** *$x, y, ...(, 1)$.*
2) If to the term t of complexity k corresponds the polynomial p then to the term $\neg t$ ***corresponds the polynomial*** *$1 - p$.*
3) If to the terms t and τ of complexity no greater than k correspond the polynomials p and q then to the term $t \cdot \tau$ ***corresponds the polynomial*** *$p \cdot q$. (We shall omit the symbol $\cdot$ of multiplication in polynomials when there is no danger of confusion).*

Since the values of any terms are equal to the values of the corresponding polynomials then terms are equal in A (and A_1) iff the corresponding polynomials are equal.

We define new notions which are interest for independent study, and are necessary for our main goal.

Definition 3. *Two terms t and τ are said* ***trivially equal terms in A*** *(denote this as $t \cong \tau$) if they can be derived from each other by chain of substitutions using only equations $\neg(\neg(t)) = t$, $t_1 \cdot t_2 = t_2 \cdot t_1$ and $(t_1 \cdot t_2) \cdot t_3 = t_1 \cdot (t_2 \cdot t_3)$.*

Definition 4. *Two terms t and τ are said* ***1-trivially equal terms in*** *A_1 (denote this as $t \cong_1 \tau$) if they can be derived from each other by chain of substitutions using only equations $\neg(\neg(t)) = t$, $t_1 \cdot t_2 = t_2 \cdot t_1$, $(t_1 \cdot t_2) \cdot t_3 = t_1 \cdot (t_2 \cdot t_3)$, $t_1 \cdot 1 = t_1$ and $t_1 \cdot \neg 1 = \neg 1$.*

Remark 1. *Note that defined relations is reflexive, symmetric and transitive.*

Example 1. *The terms* $\neg(x \cdot \neg(y \cdot x \cdot \neg(y)))$ *and* $\neg(x \cdot y) \cdot \neg(x \cdot \neg(y))$ *are equal in* A *and* A_1 *but they are not trivially equal and not 1-trivially equal.*

They are equal since correspond to the same polynomial $1 - x + x^2y - x^2y^2$. However using trivial equations from the first term it is possible to derive only terms with an odd number of negations at the head; but the second term begins by an even number (0) of these negations.

Remark 2. *Further in this paper we omit the symbol* $\cdot$ *of term product and brackets when there is no danger of confusion. We write* $\neg^k$ *instead* k *successive negations.*

Example 2. *The following two terms are equal in* A *and* A_1 *but not trivially equal and not 1-trivially equal:*

$$\begin{aligned}&\neg(x_1\neg(x_2...\neg(x_{n-1}\neg(x_nx_1\neg(x_2...\neg(x_{n-1}\neg(x_n))...)) = \\ &\neg(x_1x_2...x_n)\neg(x_1\neg(x_2...\neg(x_{n-1}\neg(x_n))...)\end{aligned} \tag{1}$$

where n *is any positive even number.*

They are equal since they correspond to the same polynomial $1 - x_1 + x_1x_2 - ... - x_1x_2...x_{n-1} + x_1^2x_2...x_n - x_1^2x_2^2x_3...x_n + ... - x_1^2x_2^2x_3^2...x_n^2$. However they are not trivially equal for the same reason as in the Example 1.

Definition 5. *A term* t *called* ***trivial (1-trivial)*** *iff any term equal to it in* A (A_1) *is trivially (1-trivially) equal to it.*

Lemma 1. *The term* t *is trivial if and only if the term* $\neg t$ *is trivial.*

Lemma 2. *If the term* $t(x_1, x_2, ..., x_n)$ *is trivial then the term* τ *derived from* t *by changing one variable into its negation* $(\tau(x_1, x_2, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n) \equiv t(x_1, x_2, ..., x_{i-1}, \neg x_i, x_{i+1}, ..., x_n))$ *is also trivial.*

Lemma 3. *The terms of the forms (2) and (3) are trivial.*

$$\neg(x_1x_2...x_{k_1}\neg(y_1y_2...y_{k_2}...\neg(z_1z_2...z_{k_n})...)), \tag{2}$$

$$x_1x_2...x_{k_1}\neg(y_1y_2...y_{k_2}...\neg(z_1z_2...z_{k_n})...). \tag{3}$$

3 Simplifications of terms in algebra A_1

Definition 6. *A term* $S(t)$ *derived from a term* t *of algebra* A_1 *by reducing its subterms by the rules:* $\neg(\neg\tau) = \tau$, $\tau \cdot 1 = \tau$, $1 \cdot \tau = \tau$, $\tau \cdot \neg 1 = \neg 1$, $\neg 1 \cdot \tau = \neg 1$ *is a* ***simplification of a term*** t*. Reductions are produced in any acceptable order. Simplification of a term is minimal that is not further reducible term.*

Remark 3. *For any term t its simplification $S(t) \equiv 1$ or $S(t) \equiv \neg 1$ or $S(t)$ does not contain 1, since $S(t)$ does not contain double negatives, and* 1 *and* $\neg 1$ *may not occur as factors. Therefore, specifying special cases $S(t) \equiv 1$ and $S(t) \equiv \neg 1$, we can consider any simplification $S(t)$ as a term of algebra A.*

Proposition 1. *For any term t of algebra A_1 its simplification $S(t)$ is equal to the term t.*

Proof. Note that the simplification rules (from the Definition 6) do not change the value of the corresponding polynomial. And since the corresponding polynomials for the terms t and $S(t)$ are identical, then these terms are equal. □

Theorem 1. *For any term t of algebra A_1 its simplification $S(t)$ is uniquely determined, ie the result does not depend on the order of reduction.*

Proof. We prove the theorem by induction on the complexity of the term t.

The induction base is obvious: for the terms x and 1 their simplification $S(x) \equiv x$ and $S(1) \equiv 1$ are uniquely determined, since none of the reduction rules are acceptable for them.

Suppose that the theorem is true for the terms whose complexity does not exceed k. We will prove that it is also true for any term t, whose complexity is equal to $k + 1$.

There are two cases of forming the last step of term t:
(i) $t \equiv t_1 \cdot t_2$,
(ii) $t \equiv \neg t_1$,
where the complexity of the subterms t_1 and t_2 do not exceed k.

In the first case simplification made in subterms t_1 and t_2 separately, and possibly between them. If reductions occur only within subterms t_1 and t_2 (this subcase occurs if the subterm t_1 and t_2 do not correspond to the polynomials 1 and 0), then by the induction hypothesis, the order of reduction of the term t does not affect the final result, which is equal to $S(t) \equiv S(t_1) \cdot S(t_2)$.

The reduction between subterms t_1 and t_2 can occur only if one of these subterms correspond to a polynomial of 1 or 0. If any of these subterms corresponds to a polynomial 0, then $S(t) \equiv \neg 1$. If any of these subterms corresponds to a polynomial 1 (without loss of generality, let $S(t_1) \equiv 1$), then $S(t) \equiv S(t_2)$, and by the induction hypothesis, $S(t_2)$ is uniquely defined.

If in the second case the various simplifications did not arise reduction of external negation, then, by induction hypothesis, $S(t)$ is uniquely determined by $S(t) \equiv \neg S(t_1)$. Reduction of external negation can occur only if at some step of simplification instead a subterm t_1 occur a subterm $\neg t'$. By the induction hypothesis, simplifications $S(t_1)$ and $S(t')$ are uniquely defined, and $S(t_1) \equiv \neg S(t')$ or $S(t') \equiv \neg S(t_1)$. In both cases, $S(t) \equiv S(t')$. □

Example 3. *The terms* 1 *and* $\neg 1$ *are 1-trivial.*

Proof. Suppose that there exists a term t which is equal to the term 1 ($t = 1$), then by Proposition 1, $S(t) = S(1) = 1$. Since the simplification rules are derivable from the 1-trivial transformations (but not vice versa), then $t \cong_1 1$. A proof is the same in the case $t = \neg 1$. □

Theorem 2. *Simplifications of arbitrary terms t and τ have the following properties:*
(i) $t \cong_1 \tau$ if and only if $S(t) \cong S(\tau)$ in the algebra of A (in this case we assume that trivial equations $1 \cong 1$ and $\neg 1 \cong \neg 1$ are valid in the algebra A);
(ii) a term t is 1-trivial if and only if the term $S(t)$ is trivial in the algebra A (in this case we assume that terms 1 and $\neg 1$ are trivial in the algebra A).

Proof. (i) Necessity. Let $t \cong_1 \tau$. If $t = \tau = 1$ or $t = \tau = \neg 1$ then $S(t) \cong S(\tau)$ since we assumed $1 \cong 1$ and $\neg 1 \cong \neg 1$ in the algebra A. In general case there is a chain of identities $t \equiv \tau_0 \cong_1 \tau_1 \cong_1 ... \cong_1 \tau_m \equiv \tau$, at each step using only the equations from Definition 4. Then simplifications of the neighboring terms τ_i and τ_{i+1} in the chain will be trivially equal in the algebra A. Indeed, the equations $\neg(\neg(t)) = t$, $t_1 \cdot 1 = t_1$ and $t_1 \cdot \neg 1 = \neg 1$ are the simplification rules themselves. If $S(\tau_i)$ and $S(\tau_{i+1})$ are different terms then between the terms τ_i and τ_{i+1} was applied either commutativity or associativity rule. Then $S(\tau_i) \cong S(\tau_{i+1})$ for any i. Therefore $S(t) \cong S(\tau)$ in the algebra of A.

(i) Sufficiency. Let $S(t) \cong S(\tau)$ in the algebra A. If $t = \tau = 1$ or $t = \tau = \neg 1$ then proof is obvious since terms 1 and $\neg 1$ are 1-trivial. In the general case from $S(t) \cong S(\tau)$ follows $S(t) \cong_1 S(\tau)$ since the rules of 1-trivial equations contain all the rules of trivial equations. Since simplifications rules are deduced from the 1-trivial transformations then $t \cong_1 S(t) \cong_1 S(\tau) \cong_1 \tau$.

(ii) Obviously follows from definitions and (i). □

Corollary 1. *If a term $S(t)$ is trivially equal to the term of the form (2) and (3), then this term $S(t)$ is trivial in the algebra A (because of Lemma 3). Therefore, because of (*ii*), a term t is 1-trivial.*

Theorem 3. *Any base of valid equations in the algebra A_1 is infinite.*

Proof. If the algebra A_1 has a finite base of equations we can assume that it also contains 1-trivial equations which express the rules of commutativity, associativity and the simplification rules $\neg(\neg(t)) = t$, $t_1 \cdot 1 = t_1$ and $t_1 \cdot \neg 1 = \neg 1$. Then all the terms of the base (except the simplification rules) can be simplified by Definition 6, and the resulting finite set of equations will remain the base of the equations of the algebra A_1. In the following text we shall use a simplified base.

For a finite base there exists a positive even integer n which restricts the number of variables in the base equations. Consider the equation (1) from the Example 2. Then for this equation it is possible to construct a chain of equations $t \equiv t_0 = t_1 = ... = t_k \equiv \tau$, where at each step used a base transformation, the term t coincides with the left part, the term τ coincides with the right side of (1). Then the following chain of equations are valid $t \equiv S(t) \equiv S(t_0) = S(t_1) = ... = S(t_k) \equiv S(\tau) \equiv \tau$. Since the simplifications $S(t)$ and $S(\tau)$ are not trivially equal in the algebra A then there exists an index i such that $t \equiv S(t) \equiv S(t_0) \cong S(t_1) \cong ... \cong S(t_i) \ncong S(t_{i+1})$. Then, by the Theorem 2, $t \equiv t_0 \cong_1 t_1 \cong_1 ... \cong_1 t_i \ncong_1 t_{i+1}$.

Suppose that in step i it was applied a base transformation $b(\theta_1, \theta_2, ..., \theta_k) = \beta(\theta_1, \theta_2, ..., \theta_k)$, where $k < n$ and $\theta_1, \theta_2, ..., \theta_k$ are subterms. Obviously, this equation is not 1-trivial $b(\theta_1, \theta_2, ..., \theta_k) \ncong_1 \beta(\theta_1, \theta_2, ..., \theta_k)$. Note also that in the simplification procedure $S(t_i)$, by Theorem 1, we can first simplify $S(b(\theta_1, \theta_2, ..., \theta_k))$. Since $t_i \ncong_1 t_{i+1}$ and $S(t_i) \ncong_1 S(t_{i+1})$ the occurrence of the term $S(b(\theta_1, \theta_2, ..., \theta_k))$ in the term $S(t_i)$ is not reduced. It is possible prove that each variable y_j in the term $S(b(y_1, y_2, ..., y_k))$ occurs no more than once and the term $S(b(y_1, y_2, ..., y_k))$ is trivially equal to the term of the form (2) or (3). Then this term $S(b(y_1, y_2, ..., y_k))$ is trivial in the algebra A (because of Lemma 3), therefore it is 1-trivial. Contradiction.

□

Remark 4. *In fact we proved a stronger statement: the set of all equations valid in the algebra A_1 has not any base in a finite number of variables.*

Indeed, in the proof of Theorem we used only the fact that there exists a restriction on the number of variables of the base equations and not used finite or infinite the base of equations.

References

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