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Sylow p-subgroups of commutative group algebras of finite abelian p-groups

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Abstract. Let RG be the group algebra of a finite abelian p-group G over a direct product R of finitely many commutative indecomposable rings with identities. Suppose that $V(RG)$ is the group of normalized units in RG and $S(RG)$ is the Sylow p-subgroup of $V(RG)$. In the present paper we establish the structure of $S(RG)$ when p is an invertible element in R. This investigation extends a result of Mollov (Zbl 0655.16004) who gives a description, up to isomorphism, of the torsion subgroup of $V(RG)$, when R is a field of characteristic different from p.

1 Introduction

Let G be a finite abelian p-group. Mollov [2] establishes the structure of the torsion subgroup of $V(RG)$ when R is a field of characteristic different from p. The present article gives a description of $S(RG)$, up to isomorphism, when R is a direct product of commutative indecomposable rings with identities such that p is invertible in R and it is an announce of results of our paper which is accepted for a publication in C. R. Acad. Bulgar. Sci. (Kuneva, Mollov and Nachev [1]).

2 Preliminary results

Let R^* be the unit group of a ring R and let p be a prime. We denote the p-component $(R^*)_p$ of R^* by R_p, that is $(R^*)_p = R_p$. Let $\alpha \in L$ be an algebraic element over the ring R and α be a root of a polynomial $f(x) \in R[x]$ of degree n. We say that $f(x)$ is a minimal polynomial of α if α is not a root of a polynomial over R which degree is less than n. We denote by $R[\alpha]$ the intersection of all subrings of L containing R and α.

Definition. A ring R of characteristic different from the prime p is called *a ring of the first kind with respect to* p, if there exists a natural $j, j \geq 2$ such that $R[\varepsilon_j] \neq R[\varepsilon_{j+1}]$. In the contrary R is called *a ring of the second kind with respect to* p.

This definition implies immediately that if R is a ring of the second kind with respect to 2, then $R[\varepsilon_2] = R[\varepsilon_j]$ for every natural $j \geq 2$.

Nachev ([6], Corollary 5.2) proves the following result (slightly modified).

Theorem 2.1. *Let R be a commutative indecomposable ring with 1 and the prime p be invertible in R. If R is a ring of the first kind with respect to p, then there exists $i \in \mathbb{N}$, such that if $p \neq 2$, then*

$$R[\varepsilon_1] = R[\varepsilon_2] = ... = R[\varepsilon_i] \neq R[\varepsilon_{i+1}] \neq ...$$

and if $p = 2$, then

$$R[\varepsilon_2] = R[\varepsilon_3]... = R[\varepsilon_i] \neq R[\varepsilon_{i+1}] \neq ...;$$

If R is a ring of the second kind with respect to p and $p \neq 2$, then $R[\varepsilon_1] = R[\varepsilon_j]$ for every $j \in \mathbb{N}$.

When R is a ring of the first kind with respect to prime p, then the number i, defined in the last theorem, is called *a constant of the ring R with respect to p.*

Let η_n be a fixed root of a monic indecomposable divisor of the cyclotomic polynomial $\Phi_n(x)$ over R. We can note that if $n = p^k$, then $\eta_{p^k} = \varepsilon_k$. Suppose $G(d)$, $d \in \mathbb{N}$, is the number of the elements of order d in G and $a(d) = G(d)/[R[\varepsilon_d] : R]$, where $[R[\varepsilon_d] : R]$ is the dimension of the free module $R[\varepsilon_d]$ over the ring R.

3 Main results

If G is an abelian p-group and $k \in \mathbb{N}$, then we denote

$$G[p^k] = \left\{ g \in G |\ g^{p^k} = 1 \right\}.$$

Let $\coprod_n G$ and $\sum_n^{\cdot} R$, where $n \in \mathbb{N}$, denote the coproduct of n copies of G and the direct sum of n copies of R, respectively.

Theorem 3.1. *Let G be a finite abelian p-group of an exponent p^n ($n \in \mathbb{N}$), R be a commutative indecomposable ring with identity, $p \in R^*$ and let R be a ring of the second kind with respect to p.*

1) If either $p \neq 2$, or $p = 2$ and $R = R[\varepsilon_2]$, then

$$S(RG) \cong \coprod_{(|G|-1)/(R[\varepsilon_1]:R]} Z(p^\infty).$$

2) If $p = 2$ and $R \neq R[\varepsilon_2]$, then

$$S(RG) \cong \coprod_{|G[2]|-1} Z(2) \times \coprod_{|G \setminus G[2]|/2} Z(2^\infty).$$

For the proof we use Theorem 2.1 and the following three results.

Theorem A (Mollov and Nachev ([3], Remark 4.5)). *Let G be a finite abelian group of exponent n and let R be a commutative indecomposable ring with identity. If n is an invertible element in R, then*

$$RG \cong \dot{\sum_{d/n}} a(d)R[\eta_d].$$

Theorem B (Nachev [6]. *Let R be a commutative indecomposable ring with 1 and the prime p be invertible in R. Then the p-component R_p of the unit group R^* is a cocyclic group.*

Theorem C (Nachev [5]). *Let R be a commutative indecomposable ring with identity and α be an algebraic element over R, such that its minimal polynomial over R is monic and indecomposable. Then $R[\alpha]$ is indecomposable.*

Theorem 3.2. *Let G be a finite abelian p group of exponent p^n ($n \in \mathbb{N}$), R be a commutative indecomposable ring with identity, $p \in R^*$ and let R be a ring of the first kind with respect to p with constant i with respect to p.*

1) If either $p \neq 2$, or $p = 2$ and $R = R[\varepsilon_2]$, then

$$S(RG) \cong \coprod_{\delta_i} Z(p^i) \times \coprod_{k=i+1}^{n} \coprod_{\delta_k} Z(p^k),$$

$\delta_i = (|G[p^i]) - 1)/[R[\varepsilon_i] : R]$, $\delta_k = |G[p^k] \setminus G[p^{k-1}]|/[R[\varepsilon_k] : R]$, $k = i + 1, ..., n$.

2) If $p = 2$ and $R \neq R[\varepsilon_2]$, then

$$S(RG) \cong \coprod_{\delta_1} Z(2) \times \coprod_{\delta_i} Z(2^i) \times \coprod_{k=i+1}^{n} \coprod_{\delta_k} Z(2^k),$$

$\delta_1 = |G[2]| - 1$, $\delta_i = |G[2^i] \setminus G[2]|/[R[\varepsilon_i] : R]$, $\delta_k = |G[2^k] \setminus G[2^{k-1}]|/[R[\varepsilon_k] : R]$, $k = i + 1, ..., n$.

The proof is obtained as the proof of Theorem 3.1.

Theorem 3.3. *Let G be a finite abelian p-group and let $R = \prod_{i \in \mathbb{I}} R_i$, where R_i are commutative indecomposable rings with identities such that p is an invertible element in R. Then*

$$S(RG) \cong (\prod_{i \in \mathbb{I}} S(R_iG))_p.$$

In particular if $R = \prod_{i=1}^{n} R_i$*, then*

$$S(RG) \cong \prod_{i=1}^{n} S(R_iG).$$

The description of $S(R_iG)$ *is given by Theorems 3.1 and 3.2.*

The proof is directly obtained by the following proposition.

Preposition (Mollov and Nachev [4]). *If* G *is a finite abelian group and* R_i*,* $i \in \mathbb{I}$*, are commutative rings with identities, then*

$$(\prod_{i\in\mathbb{I}} R_i)G \cong \prod_{i\in\mathbb{I}} R_iG.$$

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