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The non-existence of Tuscan-2 squares of order 9 [1]

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Abstract. In this paper we prove the non-existence of Tuscan-2 squares of order 9.

1 Introduction

Tuscan squares are introduced in [1].

Definition 1. *An $r \times n$ Tuscan-k rectangle has r rows and n columns such that*
(1) each row is a permutation of the n symbols
(2) for any two distinct symbols a and b, and for each m from 1 to k, there is at most one row in which b is m steps to the right of a.

Definition 2. *A Tuscan-k square of order n is an $n \times n$ Tuscan-k rectangle.*

A Tuscan square is in standard form when the top row and the left-most column contain the symbols in natural order.

Example 1. *Tuscan-2 square of order 8.*

1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	2	7	1	8	4	6	5
4	1	7	5	3	8	6	2
5	8	1	4	7	2	6	3
6	1	5	2	4	8	3	7
7	4	2	8	5	1	3	6
8	2	5	7	6	4	3	1

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2 Preliminaries

The number of standard form Tuscan-k squares of order $n \leq 13$ is presented in the next table:

$k \backslash n$	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0	1	0	736	466144	$\geq 3 \cdot 10^7$	$\geq 3 \cdot 10^7$	≥ 72	≥ 1	≥ 72	≥ 1
2		0	1	0	1	0	6	?	≥ 2	?	≥ 2	?
3			1	0	1	0	0	0	≥ 1	?	≥ 1	?
4				0	1	0	0	0	≥ 1	?	≥ 1	?
5					1	0	0	0	≥ 1	?	≥ 1	?
6						0	0	0	≥ 1	?	≥ 1	?
7							0	0	≥ 1	?	≥ 1	?
8								0	1	?	≥ 1	?
9									1	0	≥ 1	?
10										0	≥ 1	?
11											≥ 1	?
12												?

Thus the naturally arising question is if there are Tuscan-2 squares of order 9.

3 The approach

We try to construct a Tuscan-2 square of order 9 in standard form.

Definition 3. *We say that two permutations of the numbers 1,2,...,n are compatible if they form* $2 \times n$ *Tuscan-2 rectangle.*

First we generate all permutations of 1,2,...,9, which are compatible with the permutation 1,2,3,4,5,6,7,8,9.

It turns out that there are exactly 56459 such permutations.

Now we consider an undirected graph G with vertices these 56459 permutations. There is an edge between two vertices (and the vertices are said to be adjacent) iff the corresponding permutations are compatible.

It turns out that the graph G has 203 140 075 edges.

Definition 4. *A clique in G is a subset of vertices all of which are adjacent to each other.*

In this way the problem for the existence of a Tuscan-2 square of order 9 is converted to the problem for the existence of a clique of size 8 in the graph G.

We use the Cliquer program [2] which finds cliques if an undirected graph.

In fact the graph G is quite large but after a long time of computations the Cliquer program completed successfully. The result obtained is that there is no clique of size 8 in the graph G.

Consequently the following theorem holds:

Theorem 1. *There is no Tuscan-2 square of size 9.*

Note. Using the same approach we generate all six Tuscan-2 squares of order 8.

References

[1] S.W. Golomb and H. Taylor, Tuscan squares – A new family of combinatorial designs, *Ars Combin.*, **20B**, 115–132, 1985.

[2] S. Niskanen and P.R.J. Ostergard, Cliquer User's Guide, Version 1.0, Communications Laboratory, Helsinki University of Technology, Espoo, Finland, Tech. Rep. T48, 2003.