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# Combinatorial generalizations of Jung's theorem[1]

Arseniy Akopyan akopjan@gmail.com
Institute for Information Transmission Problems, Russian Academy of Sciences and Yaroslavl' State University.

**Abstract.** We consider combinatorial generalizations of Jung's theorem on covering the set with unit diameter by a ball. We prove "fractional" and "colorful" version of the theorem.

Well-known Jung's theorem state that any set with diameter 1 in $\mathbb{R}^d$ can be covered by a ball with radius $R_d = \sqrt{\frac{d}{2(d+1)}}$ (see [1]).

The proof of this theorem is based on Helly's theorem:

**Theorem 1** (Helly's theorem)**.** *Let $\mathscr{P}$ be a family of convex compacts in $\mathbb{R}^d$ such that intersection of any $d+1$ of them is not empty, than intersection of all compacts from $\mathscr{P}$ is not empty.*

Helly's theorem has many generalizations. M. Katchalski and A. Liu in 1979 [3] proved "fractional" version of Helly's theorem and G. Kalai in 1984 [2] gave a strongest version of it. L. Lovász in 1979 suggested a "colorful" version of Helly's theorem. We give analogues generalizations of Jung's theorem.

**Theorem 2** (Fraction version of Jung's theorem)**.** *For every $d \geq 1$ and every $\alpha \in (0, 1]$ there exists a $\beta = \beta(d, \alpha) > 0$ with the following property. Let $\mathcal{V}$ be a $n$-point set in $\mathbb{R}^d$ such that for at least $\alpha C_n^2$ of pairs $\{x, y\}$ $(x, y \in \mathcal{V})$ distance between $x$ and $y$ less than $1$. Then there exists a ball with radius $R_d$, which covers $\beta n$ points of $\mathcal{V}$. And $\beta \to 1$ as $\alpha \to 1$.*

We will use the following definition,

**Definition 1.** *Call two nonempty sets $\mathcal{V}_1$ and $\mathcal{V}_2$ close, if for any points $x \in \mathcal{V}_1$ and $y \in \mathcal{V}_2$, the distance between $x$ and $y$ is not greater than $1$.*

It is easy to see that if two close sets $\mathcal{V}_1$ and $\mathcal{V}_2$ are given, diameter of each of them is not greater than 2. Moreover, the following theorem holds.

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**Theorem 3.** *Union of several pairwise close sets in* $\mathbb{R}^d$ *can be covered by a ball of radius* 1*.*

It is clear that the diameter of the cover ball in this theorem could not be decreased. The following two question have sense.

Suppose a family of pairwise close sets $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ in $\mathbb{R}^d$ is given.

1. What is minimal $R$, such that at least one of the set $\mathcal{V}_i$ can be covered by a ball with radius $R$.

2. What is minimal $D$, such that at least one of the set $\mathcal{V}_i$ has a diameter not greater than $D$.

**Theorem 4.** *Let* $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ *be pairwise close sets in* $\mathbb{R}^d$*. Then one of the set* $\mathcal{V}_i$ *can be covered by a ball with radius* $R$*.*

$$R = \tfrac{1}{\sqrt{2}} \text{ if } n \le d;$$
$$R = R_d = \sqrt{\tfrac{d}{2(d+1)}} \text{ if } n > d.$$

Through $D_d(n)$ we denote the minimal diameter of optimal spherical antipodal code of cardinality $2n$ on the unit sphere $\mathbb{S}^{d-1}$.

**Theorem 5.** *Let* $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ *be pairwise close sets in* $\mathbb{R}^d$*. Then one of the set* $\mathcal{V}_i$ *has diameter not greater than*

$$D = \frac{2}{\sqrt{4 - D_d(n)^2}}.$$

# References


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