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An improved algorithm for proving nonexistence of small spherical designs[1]

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Abstract. We develop further our algorithm for obtaining nonexistence results for spherical designs of odd strength and relatively small cardinalities. We show how a procedure around a special triple of points in the design can be organized to result in stronger bounds on the extreme inner products of some points. New nonexistence results appear either in small dimensions and in the asymptotic process when the strength is fixed and the dimension and cardinality tend to infinity in certain relation.

1 Introduction

A spherical τ-design [8] is a spherical code $C \subset \mathbf{S}^{n-1}$ such that for every point $x \in C$ and for every real polynomial $f(t)$ of degree at most τ, the equality

$$\sum_{x \in C \setminus \{y\}} f(\langle t_i(x) \rangle) = f_0|C| - f(1), \qquad (1)$$

holds, where the number f_0 is the first coefficient in the Gegenbauer expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ and $t_1(x) \leq t_2(x) \leq \cdots \leq t_{|C|-1}(x)$ are all inner products of $x \in C$ with all other points of C. The number τ is called strength of C.

Polynomial techniques use suitable polynomials in (1) for obtaining bounds on some inner products. Restrictions on the structure of spherical designs via polynomial techniques were described in 1997 by Fazekas-Levenshtein [9] (see also [10]) and proved to work for nonexistence results by Boyvalenkov-Danev-Nikova [5] (see also [1, 2, 4]).

We study the smallest possible odd size of a τ-design on $C \subset \mathbb{S}^{n-1}$ for fixed $n \geq 3$ and odd $\tau \geq 3$ in terms of lower bounds on

$$B_{\text{odd}}(n, \tau) = \min\{|C| : C \subset \mathbb{S}^{n-1} \text{ is a } \tau\text{-design with odd cardinality } |C|\}.$$

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2 Some preliminaries

Let the integers $n \geq 3$, odd $\tau = 2k-1 \geq 3$, and odd M be fixed and let $C \in \mathbb{S}^{n-1}$ be a spherical τ-design of odd size $|C| = M$. Then there exist (cf. [10, Section 4], [4]) uniquely determined real numbers $-1 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < 1$ and positive $\rho_0, \rho_1, \ldots, \rho_{k-1}$ such that the equality

$$f_0 = \frac{f(1)}{M} + \sum_{i=0}^{k-1} \rho_i f(\alpha_i) \tag{2}$$

holds for every real polynomial $f(t)$ of degree at most $2k-1$. The numbers α_i, $i = 0, 1, \ldots, k-1$, are the roots of certain equation involving Jacobi polynomials. We denote $g(t) = \prod_{i=1}^{k-1} (t - \alpha_i)^2 = \sum_{i=0}^{2k-2} g_i P_i^{(n)}(t)$ and $1 + \gamma(k-1)! := \theta$ for short. Then (2) implies that $g_0 = \rho_0 |C| g(\alpha_0)$.

Some previous results are summarized in the following theorem. We denote by $U_{\tau,i}(x)$ (respectively $L_{\tau,i}(x)$) for any upper (resp. lower) bound on the inner product $t_i(x)$, omitting x if the corresponding bound is valid for all $x \in C$. In the asymptotic, we always assume that the dimension n is large enough to have all corresponding bounds valid.

Theorem 1. *[1, 4, 6] Let $C \subset \mathbb{S}^{n-1}$ be a τ-design with odd strength $\tau = 2k-1$ and odd size $M = |C|$. Then $\rho_0 |C| \geq 2$ and:*

a) $t_1(x) \leq U_{\tau,1} = \alpha_0$ and $t_{|C|-1}(x) \geq L_{\tau,|C|-1} = \alpha_{k-1}$ hold for every point $x \in C$;

b) there exist three distinct points $x, y, z \in C$ such that $\langle x, y \rangle = t_1(x) = t_1(y)$, $\langle x, z \rangle = t_2(x) = t_1(z)$ and $\langle y, z \rangle \geq 2\alpha_0^2 - 1$;

c) if $M = \left(\frac{2}{(k-1)!} + \gamma\right) n^{k-1}$, where $\gamma > 0$ is constant, n tends to infinity, then $\alpha_i \sim 0$, for $i = 1, 2, \ldots, k-1$, $\theta\alpha_0 \sim -1$; $g(t) \sim t^{2k-2}$, $\rho_0 |C| \sim \theta^{2k-1}$ and $\rho_0 |C| g(\alpha_0) \sim \theta$.

The inequalities $2 \leq \rho_0 |C| \leq a$, $a \in \{3, 4\}$, correspond to $\gamma < \frac{\sqrt[2k-1]{a} - 1}{(k-1)!}$.

3 Different types of special triples

Let $\{x, y, z\} \subset C$ be a special triple as in Theorem 1b). If $t_2(y) > \alpha_0$ or $t_2(z) > \alpha_0$ we call such triple y-bad or z-bad, respectively. These cases are the easiest in our algorithm and we refer for details in [3, 6, 7]. Here we add an improving argument in the worse, in the sense of the analysis in [3, 6, 7], case, namely when there are no special triples which are y-bad or z-bad.

Theorem 2. *If there exist no bad triples in C then at least one of the following holds:*

(i) there exists a special triple $\{x, y, z\} \subset C$ *such that* $t_{|C|-2}(x) \geq 2\alpha_0^2 - 1$ *and* $t_{|C|-2}(z) \geq 2\alpha_0^2 - 1$*;*

(ii) there exists a point $x' \in C$ *such that* $t_3(x') \leq \alpha_0$*.*

Proof. We start with a (non-bad) special triple (x, y, z) and consider the point $v_1 \in C$ defined by $\langle y, v_1 \rangle = t_2(y)$. Then $\langle y, v_1 \rangle = t_i(v_1) = t_2(y) \leq \alpha_0$ (here $i \geq 1$) since the triple (x, y, z) is not y-bad. Now consider the point $v_2 \in C$ defined by $\langle v_1, v_2 \rangle = t_1(v_1)$ if $i \geq 2$ or by $\langle v_1, v_2 \rangle = t_2(v_1)$ if $i = 1$. In both cases we have $t_2(v_1) \leq \alpha_0$ – this is obvious in the first case and in the second one the converse inequality $\langle v_1, v_2 \rangle = t_2(y) > \alpha_0$ leads to a bad triple (y, x, v_1) (it would be what is z-bad).

It is easy to see that this procedure gives $t_2(v_j) \leq \alpha_0$ for every newly involved point $v_j \in C$. Since C is finite, we will reach at some step a point $x' \in C$ for second time. If $x' = z$ we obtain (i), otherwise x' already has two inner products which are less than or equal to α_0 and we add third such inner product, i.e. $t_3(x') \leq \alpha_0$. □

The argument in Theorem 2 can start from the point z of a special triple (x, y, z). Then the end of the procedure could be y (then (i) follows) or other point of C (then (ii) follows).

It is easy to see that the case (ii) in Theorem 2 necessarily leads to $\rho_0|C| \geq 3$. The converse inequality was imperative in [3, 6]. Overcoming this is the major improvement, given by Theorem 2.

4 New nonexistence results

For $\tau = 3$, 37 cases with $n \leq 50$ and $\rho_0|C| \leq 3$ (thus Theorem 2(i) holds in the corresponding case) were left open after [3]. Our strengthening allows to rule out 21 of them. The remaining 16 are listed here.

Theorem 3. *Let* $C \subset \mathbb{S}^{n-1}$, $3 \leq n \leq 50$, *be a spherical 3-design with odd cardinality* M*. Then* $\rho_0|C| > 3$ *with possible exceptions in 16 cases:* $(n, M) =$ $(11, 27)$, $(15, 37)$, $(20, 49)$, $(24, 59)$, $(25, 61)$, $(29, 71)$, $(30, 73)$, $(33, 81)$, $(34, 83)$, $(38, 93)$, $(39, 95)$, $(42, 103)$, $(43, 105)$, $(44, 107)$, $(47, 115)$, $(48, 117)$.

Similarly, for $\tau = 5$, our strengthened approach allows calculations in more cases, namely in dimensions $25 \leq n \leq 50$, which were not considered earlier. Here we ruled out all cases with $\rho_0|C| \leq 3$ and some cases with $\rho_0|C| > 3$ (in total, 968 cases).

In the asymptotic, we consider again separately the three cases of a bad triple, Theorem 2(i) and Theorem 2(ii) to find in each case the maximal possible γ for which all nonexistence criteria follow.

For example, we have $B_{\text{odd}}(n,3) \gtrsim 2.421n$ and $B_{\text{odd}}(n,5) \gtrsim 1.1245n^2$, compared to $B_{\text{odd}}(n,3) \gtrsim 2.3925n$ from [2] and $B_{\text{odd}}(n,5) \gtrsim 1.12286n^2$ from [6].

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