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The maximum and the minimum sizes of complete $(n,3)$-arcs in $PG(2,16)$

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Abstract. In this work we solve the packing problem for complete $(n,3)$-arcs in $PG(2,16)$, determining that the maximum size is 28 and the minimum size is 15. Both the complete $(28,3)$-arc and the complete $(15,3)$-arc are unique up to collineations.

1 Introduction

In the projective plane $PG(2,q)$ over the finite field $GF(q)$, an (n,r)-arc is a set of n points no $(r+1)$ of which are collinear containing r collinear points. An (n,r)-arc is called *complete* if it is not contained in a $(n+1,r)$-arc of the same projective plane. An $(n,2)$-arc is called n-arc. For a more detailed introduction to (n,r)-arcs and $(n,3)$-arcs see [7], [8]. The largest size of (n,r)-arcs of $PG(2,q)$ is indicated by $m_r(2,q)$. In [8] bounds for $m_r(2,q)$ and the relationship among the theory of complete (n,r)-arcs, coding theory and mathematical statistics are given. In particular for $q \geq 4$, $m_3(2,q) \leq 2q+1$ holds (see [11]).
Arcs and $(n,3)$-arcs in $PG(2,q)$ correspond to respectively MDS and NMDS codes of dimension 3 (see [6] for a more detailed introduction to NMDS codes). These types of linear codes are the best in term of minimum distance, among the linear codes with the same length and dimension. In general (n,k)-arcs in $PG(2,q)$ correspond to linear codes with Singleton defect equal to $k-2$.
In this work we solve the packing problem for complete $(n,3)$-arcs in $PG(2,16)$, determining that the maximum size is 28 and the minimum size is 15. Both the complete $(28,3)$-arc and the complete $(15,3)$-arc are unique up to collineations. These results have been obtained by computer search; Section 2 contains the description of the algorithm used, while Section 3 contains the results about the extremal $(n,3)$-arcs in $PG(2,16)$. Section 4 contains bounds on the sizes of the extremal $(n,3)$-arcs in $PG(2,17)$ and $PG(2,19)$ obtained using the same algorithm.

2 Algorithm

The base algorithm used is the same described in [9]. The algorithm uses isomorphism rejection and introduces constraints on the structure of the solution, as in this kind of problems some strategies have to be used to reduce the search space since there are many equivalent parts of the search space and a large number of copies of equivalent solutions could be found. The first constraint is based on the following theorem (see [3]).

Theorem 1. *An* $(n, 3)$*-arc* $\mathcal{K}$ *in* $PG(2, q)$*,* $n \geq \alpha + \binom{\alpha}{2}$*, contains an arc of size* $\alpha + 1$*.*

The algorithm searches for complete $(n, 3)$-arcs containing arcs of a fixed size s and, at least in principle, not containing arcs of size $s + 1$. The search is divided into three steps.

1. All the non-equivalent arcs C_s^i of a certain size s complete and non-complete are generated.

2. The classification process continues extending each arc C_s^i as $(t, 3)$-arc until it reaches the level $s + h$ (usually $h = 1, 2, 3$).

3. When all the non equivalent $(s+h, 3)$-arcs are generated, the leaves are extended to reach the desired length using a backtracking algorithm. Since the extension processes are independent of each other, it has been possible to realize a simple but efficient example of data parallelism. During the extension phase the information obtained in the classification is used to further reduce the search space.

However, when looking for $m_3(2, 16)$, the great number of leaves in the tree and of levels in the backtracking search makes impossible ending the tasks in a reasonable time. The most difficult cases are those starting from arcs of sizes 9 and 10.
The first problem of the algorithm described previously is that, searching for complete $(n, 3)$-arcs containing an $(s, 2)$-arc, during the backtracking we cannot avoid that the $(t, 3)$-arcs considered contain an arc of size greater than s. Such a $(t, 3)$-arc should not be considered since it has been examined when starting from $(s', 2)$-arcs, with $s' > s$. In order to avoid the greatest possible number of such $(t, 3)$-arcs, the following two ideas are used.

1. Before starting the backtracking algorithm, some information is computed. Let C' be the $(n, 3)$-arc to extend and C be the $(s, 2)$-arc contained in C'. The program computes all the pairs $(P, Q), P, Q \in PG(2, q) \setminus C'$ such that:

$$\exists\, R \in C \mid (C \setminus \{R\}) \cup \{P, Q\} \text{ is an } (s + 1, 2)\text{-arc.}$$

   The program collects all these pairs in a table. When adding a point P to the partial solution, all the points Q such that the pair (P, Q) is in the table are avoided.

2. The technique described in the previous point does not assure that a $(n,3)$-arc generated during the backtracking does not contain an $(s+1,2)$-arc. In fact it is possible that there exist $R_1,\ldots,R_\ell \in C$ and $P_1,\ldots,P_{\ell+1} \in PG(2,q)\setminus C'$ such that $(C\setminus\{R_1\ldots,R_\ell\})\cup\{P_1,\ldots,P_{\ell+1}\}$ is an $(s+1,2)$-arc. To reduce the number of these cases, a random control has been added. In this way it is possible to control at a certain level of the backtracking if the the partial solution contains an arc too big and therefore can be pruned.

Another idea is taken from [5]. It introduces a constraint on the structure of the solution concerning the distribution of the candidates points on the secants of the $(s,2)$-arc. This technique is more effective when searching for big $(n,3)$-arcs and it has been applied when searching for complete $(n,3)$-arcs of size $28 \leq n \leq 33$ containing an arc of size $8,9$ or 10.

3 Results

We establish the maximum and the minimum sizes of complete $(n,3)$-arcs in $PG(2,16)$ and classify such extremal $(n,3)$-arcs.

Theorem 2. *The maximum size of complete* $(n,3)$*-arcs in* $PG(2,16)$ *is* 28*. There exists a unique* $(28,3)$*-arc.*

Proof. We performed an exhaustive search of $(n,3)$-arcs in $PG(2,16)$ of size greater than 28 and we found no examples. Moreover we have an example of complete $(28,3)$-arc (see [4], [3]). An exhaustive search showed that this is the unique example of $(28,3)$-arc in $PG(2,16)$. □

This complete $(28,3)$-arc is an example of $(1,18)$-saturating set with μ-density $\delta = 1.285714$ (see [2]).

For the computer searches, we used a 3.2 Ghz Intel Exacore 16 Gb of memory.
Table 1 describes in detail the execution time of the search for $(n,3)$-arcs, $n \geq 29$, in $PG(2,16)$.

Table 1: Execution time of the search for $(n,3)$-arcs in $PG(2,16)$ with $n \geq 29$ containing an arc $\mathcal{A}$

$\lvert\mathcal{A}\rvert$	8	9	10	11	12	13	14	15	16	17
Time	2 d	22 d	20 d	4 d	2 d	1 h	11 m	2 m	<1 m	<8 s

Table 2: The complete $(15,3)$-arc

Points	ℓ_0	ℓ_1	ℓ_2	ℓ_3	G
1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 1 0 1 2 2 4 9 9 11 11 13 13 0 0 1 1 11 8 5 10 10 2 8 2 11 1 12	92	138	12	31	$\mathcal{S}_3$

The search for complete $(28,3)$-arcs in $PG(2,16)$ lasted about 132 days of total CPU time. Both searches have been performed exploiting data parallelism as described in the previous section.

In the language of coding theory, Theorem 2 can be rewritten as:

Theorem 3. *No* $[29,3,26]_{16}$*-code exists.*

The following corollary closes, for $h \leq 27$, some open cases in tables of [10]:

Corollary 4. *No* $[29+h, 3+h, 26]_{16}$*-code exists,* $h \geq 1$.

We are extending the unique $(28,3)$-arc found in order to obtain a $[29,4,25]$-NMDS code. The process is in progress (done 85%).

Conjecture 5. *There exists no* $[29,4,25]_{16}$*NMDS code.*

Theorem 6. *The minimum size of complete* $(n,3)$*-arcs in* $PG(2,16)$ *is* 15. *There exists a unique complete* $(15,3)$*-arc.*

Proof. We performed an exhaustive search of $(n,3)$-arcs in $PG(2,16)$ of size less than 15 and we found no examples (execution time: 11 days). Moreover we proved that there exists only one complete $(15,3)$-arc up to collineations (see Table 2) and it contains a $(9,2)$-arc, but not a greater arc. The classification lasted 45 days. $\square$

In Table 2 we denote $GF(16) = \{0, 1 = \alpha^0, 2 = \alpha^1, \ldots, 15 = \alpha^{14}\}$ where α is a primitive element such that $\alpha^4 + \alpha^3 + 1 = 0$. In all the tables, the columns ℓ_i indicate the number of i-secants of the $(n,3)$-arc and G indicates the description of the stabilizer group. In Table 2 we consider the stabilizer group in $P\Gamma L(3,16)$.

4 On extremal $(n,3)$-arcs in $PG(2,17)$ and $PG(2,19)$

Using the algorithms described in Section 2, we performed partial searches in $PG(2,17)$ and in $PG(2,19)$ in order to find examples and bounds on the size of complete $(n,3)$-arcs of extremal size. Some examples, obtained using a partial classification, are presented in Tables 3, 4, 5. The following theorems hold.

Table 3: $(18, 3)$-arcs in $PG(2, 17)$, containing an arc $\mathcal{A}$

$\lvert\mathcal{A}\rvert$	Points	ℓ_0	ℓ_1	ℓ_2	ℓ_3	G
9	1 0 0 1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 1 1 1 3 3 4 4 9 10 10 12 13 13 14 14 0 0 1 1 2 5 7 13 4 5 13 3 4 0 1 2 2 7	94	144	27	42	$\mathbb{Z}_1$
10	1 0 0 1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 1 1 1 2 2 3 3 10 10 12 13 15 15 16 16 0 0 1 1 5 8 4 11 7 12 4 7 9 1 0 16 8 16	97	135	36	39	$\mathbb{Z}_1$

Table 4: $(31, 3)$-arcs in $PG(2, 19)$, containing a 14-arc

Points	ℓ_0	ℓ_1	ℓ_2	ℓ_3	G
1 0 0 1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 1 1 0 1 2 2 3 4 4 5 6 6 7 7 8 8 9 10 11 11 12 15 15 16 16 17 17 18 0 0 1 1 14 5 13 5 18 7 6 11 10 3 17 16 17 6 18 12 4 4 8 12 3 8 0 1 2 9 9	101	65	90	125	$\mathbb{Z}_1$
1 0 0 1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 1 1 0 1 2 2 3 4 4 5 7 7 8 8 9 9 10 11 11 12 13 13 14 15 16 16 18 18 0 0 1 1 14 16 8 5 9 16 7 12 10 14 15 3 18 0 4 6 12 17 3 6 14 7 8 10 17 4 15	96	80	75	130	$\mathbb{Z}_1$
1 0 0 1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 1 1 0 1 2 2 3 4 4 5 5 6 6 7 8 8 9 10 11 11 12 15 15 16 16 17 17 18 0 0 1 1 14 5 13 5 18 7 6 11 10 16 3 17 16 6 18 12 4 4 8 12 3 8 0 1 2 9 9	100	68	87	126	$\mathbb{Z}_1$
1 0 0 1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 1 1 2 3 4 4 5 5 6 6 7 7 8 8 9 9 10 10 12 13 13 14 15 15 16 16 18 18 0 0 1 1 13 5 7 6 9 7 10 0 10 6 16 14 18 12 13 1 4 4 12 13 8 8 16 5 15 9 14	92	92	63	134	$\mathbb{Z}_4$

Theorem 7. *There exist no complete $(n, 3)$-arcs in $PG(2, 17)$, with $n > 28$ containing an arc of size greater than* 12*.*

Theorem 8. *There exist no complete $(n, 3)$-arcs in $PG(2, 17)$, with $n \leq 17$ containing an arc of size less than* 8*. The smallest size of complete $(n, 3)$-arcs in $PG(2, 17)$ is at most* 18*.*

Theorem 9. *The maximum size of complete $(n, 3)$-arcs in $PG(2, 19)$ is at least* 31*. There exist no complete $(n, 3)$-arcs in $PG(2, 19)$, with $n \geq 31$ containing an arc of size greater than* 14*.*

Theorem 10. *The smallest size of complete $(n, 3)$-arcs in $PG(2, 19)$ is at most* 20*.*

Proof. We have found several examples of complete $(20, 3)$-arcs (see [1]); in Table 5 we present one example containing a 10-arc. $\square$

Table 5: $(20, 3)$-arcs in $PG(2, 19)$, containing a 10-arc

Points	ℓ_0	ℓ_1	ℓ_2	ℓ_3	G
1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 1 0 1 2 2 3 3 5 5 6 8 8 12 13 15 15 17 0 0 1 1 11 9 2 5 0 8 9 10 2 4 18 11 17 8 10 18	121	170	40	50	$\mathbb{Z}_1$

References

[1] D. Bartoli, Construction and Classification of geometrical structures, *PhD Thesis*, Università degli Studi di Perugia (2012).

[2] D. Bartoli, A. A. Davydov, M. Giulietti, S. Marcugini and F. Pambianco Multiple coverings of the farthest-of points and multiple saturating sets in projective spaces, , in *Proc. XIII Int. Workshop on Algebraic and Combin. Coding Theory, ACCT2012, Pomorie, Bulgaria, 2012*, to appear.

[3] J. Bierbrauer, G. Faina, S. Marcugini and F. Pambianco, On the structure of the (n, r)-arcs in $PG(2, q)$, *Proceedings of the Tenth International Workshop on Algebraic and Combinatorial Coding Theory, Zvenigorod, Russia, 3-9 Settembre 2006*, 19-23.

[4] M. Braun, A. Kohnert and A. Wassermann, Construction of linear codes with prescribed distance, *OC05 The fourth International Workshop on Optima Codes and related Topics, PAMPOROVO, Bulgaria* (2005), 59-63.

[5] G. R. Cook, Arcs in a Finite Projective Plane, *PhD Thesis*, available on internet http://sro.sussex.ac.uk/

[6] S. Dodunekov and I. Landgev, Near-MDS codes, *Journal of Geometry* **54** (1995), 30-43.

[7] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields*, second edition, Oxford University Press, Oxford, (1998).

[8] J. W. P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory, and finite projective spaces: update 2001, *Finite Geometries, Proceedings of the Fourth Isle of Thorns Conference, A. Blokhuis, J. W. P. Hirschfeld, D. Jungnickel and J. A. Thas, Eds., Developments in Mathematics 3, Kluwer Academic Publishers, Boston,* (2000), 201-246.

[9] S. Marcugini, A. Milani and F. Pambianco, Classification of the $(n, 3)$-arcs in $PG(2, 7)$, *Journal of Geometry* **80** (2004), 179-184.

[10] MinT online database, http://mint.sbg.ac.at.

[11] J. Thas, Some results concerning $((q+1)(n-1), n)$-arcs, *J. Combin. Theory Ser. A* **19** (1975), 228-232.