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Classification of minimal 1-saturating sets in $PG(2, q)$, $q \leq 23$

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Abstract. Minimal 1−saturating sets in the projective plane $PG(2, q)$ are considered. They correspond to covering codes which can be applied to many branches of combinatorics and information theory, as data compression, compression with distortion, broadcasting in interconnection network, write-once memory and steganography (see [3] and [2]). The classification of all the minimal 1-saturating sets in $PG(2, 9)$ and $PG(2, 11)$ and the classification of minimal 1-saturating sets of the smallest size in $PG(2, q)$, $16 \leq q \leq 23$ are given. These results have been found using a computer-based exhaustive search that exploits projective equivalence properties.

1 Introduction

Let $PG(n, q)$ be the n-dimensional projective space over the Galois field $GF(q)$. For an introduction to such spaces and the geometrical objects therein, see [8] - [11].

Definition 1. *A point set S in the space $PG(n, q)$ is ϱ-saturating if ϱ is the least integer such that for any point $x \in PG(n, q)$ there exist $\varrho + 1$ points in S generating a subspace of $PG(n, q)$ in which x lies.*

Definition 2. *[14] A ϱ-saturating set of l points is called minimal if it does not contain a ϱ-saturating set of $l - 1$ points.*

A q-ary linear code with codimension r has covering radius R if every r-positional q-ary column is equal to a linear combination of R columns of a parity check matrix of this code and R is the smallest value with such property. For an introduction to coverings of vector spaces over finite fields and to the concept of code covering radius, see [3].

The points of a ϱ−saturating set in $PG(n, q)$ can be considered as columns of a parity check matrix of a q-ary linear code with codimension $n + 1$. So, in terms of the coding theory, a ϱ−saturating l-set in $PG(n, q)$ corresponds to a parity check matrix of a q-ary linear code with length l, codimension $n + 1$, and covering radius $\varrho + 1$ [4], [7], [12]. Such code is denoted by an $[l, l - (n+1)]_q(\varrho+1)$ code. Covering codes can be applied to many branches of combinatorics and information theory, as data compression, compression with distortion, broadcasting in interconnection network, write-once memory and

steganography (see [3] and [2]).
Note that a ϱ-saturating set in $PG(n,q)$, $\varrho+1 \leq n$, can generate an infinite family of ϱ-saturating sets in $PG(N,q)$ with $N = n+(\varrho+1)m$, $m = 1,2,3,\ldots$, see [3, Chapter 5.4], [4], [5, Example 6] and references therein, where such infinite families are considered as linear codes with covering radius $\varrho+1$.
In the projective plane $PG(2,q)$ over the Galois field $GF(q)$, an $n-$arc is a set of n points no 3 of which are collinear. An n-arc is called complete if it is not contained in an $(n+1)$-arc of the same projective plane. The complete arcs of $PG(2,q)$ are examples of minimal 1-saturating sets, but there are minimal 1-saturating sets that are not complete arcs. Properties of the ϱ-saturating sets in $PG(n,q)$ are presented in [6].

2 The computer search for the non-equivalent minimal $1-$saturating sets

Our goal is to determine the classification of saturating sets up to projective equivalence in $PG(2,q)$. The problem of finding non-equivalent geometrical structures is very popular in literature (see [8], [9], [10], [11]). In [13] the full classification of minimal 1-saturating sets in $PG(2,q)$, $q \leq 8$, the classification of minimal 1-saturating sets in $PG(2,q)$ of smallest size for $9 \leq q \leq 13$ and the determination of the smallest size of minimal 1-saturating sets in $PG(2,16)$ are presented.
In this work we perform an exhaustive search using a backtracking algorithm. Some strategies have to be used to reduce the search space, as in this kind of problems there are many equivalent parts of the search space and a large number of copies of equivalent solutions could be found. The program starts classifying the sets in $PG(2,q)$ containing the projective frame, until a certain size k. We only searched for minimal 1-saturating sets containing a projective frame, since the following theorem holds.

Theorem 1 *In $PG(2,q)$ there exists a unique minimal 1-saturating set not containing a projective frame. It consists of a whole line and an external point. Its stabilizer has size* $\frac{|PGL(3,q)|}{q^2(q^2+q+1)}$ $\left(or\ \frac{|P\Gamma L(3,q)|}{q^2(q^2+q+1)}\right)$.

Then the sets of size k are extended using backtracking. During the backtracking some information obtained during the classification phase is used to further prune the search space. The sets are tested for the saturating property and the minimality condition. See [1] for a detailed description.
The following tables present the results obtained. In particular we perform the classification of all the minimal 1-saturating sets in $PG(2,9)$ and $PG(2,11)$ and the classification of the minimal 1-saturating sets of the smallest size in $PG(2,q)$ with $16 \leq q \leq 23$.
We found no examples of minimal $q+2$-saturating sets in $PG(2,9)$ and $PG(2,11)$ containing the projective frame and then the unique example is that one described above. In the following table we describe the results obtained, in particular the type of the stabilizer group of the minimal 1-saturating sets of size

k. With the symbol G_i we denote a group of order i; for the other symbols we refer to [15]. If q is prime we consider stabilizer groups in $PGL(3,q)$, otherwise in $P\Gamma L(3,q)$. When complete arcs exist, their number is indicated in bold font.

$PG(2,9)$	$k=6$	G_{120}: **1**			
	$k=7$	$\mathbb{Z}_4$: 1	G_{42}: **1**	G_{120}: 1	
	$k=8$	$\mathbb{Z}_1$: 88	$\mathbb{Z}_2$: 52	$\mathbb{Z}_2\times\mathbb{Z}_2$: 11	$\mathcal{S}_3$: 1
		$\mathbb{Z}_2\times\mathbb{Z}_4$: 1	$\mathcal{D}_4$: 1	$\mathcal{D}_6$: 3	G_{16}: 1+**1**
		G_{24}: 2	G_{48}: 1		
	$k=9$	$\mathbb{Z}_1$: 667	$\mathbb{Z}_2$: 87	$\mathbb{Z}_3$:9	$\mathbb{Z}_2\times\mathbb{Z}_2$: 4
		$\mathcal{S}_3$: 2	$\mathcal{D}_4$: 1	$\mathcal{D}_6$: 1	G_{16}: 1
		G_{48}: 1			
	$k=10$	$\mathbb{Z}_1$: 58	$\mathbb{Z}_2$: 22	$\mathbb{Z}_4$:5	$\mathbb{Z}_2\times\mathbb{Z}_2$: 4
		$\mathcal{D}_4$: 2	G_{16}: 1	G_{20}: 1	G_{32}: 1+**1**
		G_{1440}: 1			
	$k=11$	G_{11520}: 1			
$PG(2,11)$	$k=7$	$\mathbb{Z}_7\rtimes\mathbb{Z}_3$: **1**			
	$k=8$	$\mathbb{Z}_1$: 22	$\mathbb{Z}_2$: 26+**5**	$\mathbb{Z}_2\times\mathbb{Z}_2$: 2+**1**	$\mathcal{D}_4$: 1 + **1**
		$\mathcal{D}_5$: **1**	G_{16}: **1**		
	$k=9$	$\mathbb{Z}_1$: 10686	$\mathbb{Z}_2$: 265+**1**	$\mathbb{Z}_3$: 40 +**1**	$\mathbb{Z}_4$: 2
		$\mathbb{Z}_2\times\mathbb{Z}_2$: 3	$\mathcal{S}_3$: 10+**1**	$\mathbb{Z}_{10}$: 1	$\mathcal{Q}_6$: 1
	$k=10$	$\mathbb{Z}_1$: 115731	$\mathbb{Z}_2$: 1332	$\mathbb{Z}_3$: 31	$\mathbb{Z}_4$: 15
		$\mathbb{Z}_2\times\mathbb{Z}_2$: 13	$\mathbb{Z}_5$: 2	$\mathcal{S}_3$: 8	$\mathcal{D}_4$: 2
			$\mathcal{D}_5$: 2	$\mathbb{Z}_{10}$: 1$\mathcal{Q}_6$:1	G_{60}:**1**
	$k=11$	$\mathbb{Z}_1$: 30802	$\mathbb{Z}_2$: 147	$\mathbb{Z}_4$: 1	$\mathbb{Z}_2\times\mathbb{Z}_2$: 3
		$\mathcal{D}_4$: 3			
	$k=12$	$\mathbb{Z}_1$: 119	$\mathbb{Z}_2$: 7	$\mathbb{Z}_3$: 5	$\mathcal{S}_3$: 1
		$\mathcal{Q}_6$: 1	G_{20}: 1	G_{1320}: **1**	
	$k=13$	G_{13200}: 1			
$q=16$	$k=9$	$\mathbb{Z}_3$: **1**	$\mathbb{Z}_6$: **1**	$\mathcal{D}_6$: 1	$\mathcal{G}_{54}$: 1
	$k=10$	$\mathbb{Z}_1$: 7744+**342**	$\mathbb{Z}_2$: 699+**130**	$\mathbb{Z}_3$: 3	$\mathbb{Z}_4$: 12+**8**
		$\mathbb{Z}_2\times\mathbb{Z}_2$: 27+**4**	$\mathbb{Z}_6$: 2	$\mathcal{S}_3$: 4+**3**	$\mathcal{D}_4$: 8
		$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$: 18+**10**	$\mathcal{Q}_6$: 1	$\mathbb{Z}_4\times\mathbb{Z}_4$: 1	G_{16}: 4+**3**
		G_{20}: **1**	G_{24}: 1	G_{32}: 1	G_{48}: 1
$q=17$	$k=10$	$\mathbb{Z}_1$: 2591+**341**	$\mathbb{Z}_2$: 460+**179**	$\mathbb{Z}_3$: 8+**10**	$\mathbb{Z}_4$: 4+**7**
		$\mathbb{Z}_2\times\mathbb{Z}_2$: 5+**8**	$\mathcal{S}_3$: 7+**9**	$\mathcal{D}_4$: 4	$\mathcal{Q}_4$: **1**
		$\mathcal{Q}_6$: **2**	G_{16}: 1+**1**	G_{18}: **1**	G_{24}: **1**
$q=19$	$k=10$	$\mathbb{Z}_1$: 1+**1**	$\mathbb{Z}_2$: 6+**18**	$\mathbb{Z}_3$: **1**	$\mathbb{Z}_4$: **1**
		$\mathbb{Z}_2\times\mathbb{Z}_2$: **2**	$\mathcal{S}_3$: **2**	$\mathcal{D}_5$: **2**	$\mathcal{Q}_6$: **1**
		G_{60}: **1**			
$q=23$	$k=10$	$\mathcal{S}_3$: **1**			

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