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New upper bounds on the smallest size of a complete arc in the plane $PG(2, q)$

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Abstract. In the projective planes $PG(2, q)$, more than 800 new small complete arcs are obtained for $q \leq 10111$ and for $q \in G$ where G is a set of about 40 values in the range $11003 \ldots 44017$; also, $2^{18} \in G$. This implies new upper bounds on the smallest size $t_2(2, q)$ of a complete arc in $PG(2, q)$. From the new bounds it follows that the relation $t_2(2, q) < 4.5\sqrt{q}$ holds for $q \leq 2647$ and $q = 2659, 2663, 2683, 2693, 2753, 2801$. Also, we have $t_2(2, q) < 4.8\sqrt{q}$ for $q \leq 5419$ and $q = 5441, 5443, 5471, 5483, 5501, 5521$. Finally, it holds that $t_2(2, q) < 5\sqrt{q}$ for $q \leq 9497$ and $q = 9539, 9587, 9613, 9649, 9689, 9973$. Using the new arcs we show that, for $109 \leq q \leq 10111$ and $q \in G$, the inequality $t_2(2, q) < \sqrt{q}\ln^{0.73} q$ is true. Moreover, our results allow us to conjecture that the estimate $t_2(2, q) < \sqrt{q}\ln^{0.73} q$ holds for all $q \geq 109$. The new upper bounds are obtained by finding new small complete arcs in $PG(2, q)$ with the help of a computer search using randomized greedy algorithms.

Let $PG(2, q)$ be the projective plane over the Galois field F_q. An n-arc is a set of n points no three of which are collinear. An n-arc is called complete if it is not contained in an $(n + 1)$-arc of $PG(2, q)$.

In [3] the relationship among the theory of n-arcs, coding theory and mathematical statistics is presented. In particular, a complete arc in a plane $PG(2, q)$, points of which are treated as 3-dimensional q-ary columns, defines a parity check matrix of a q-ary linear code with codimension 3, Hamming distance 4, and covering radius 2. Arcs can be interpreted as linear maximum distance separable (MDS) codes [3] and they are related to optimal coverings arrays and to superregular matrices.

One of the most important problems in the study of projective planes, which is also of interest in coding theory, is finding of the smallest size $t_2(2, q)$ of a complete arc in $PG(2, q)$. This is a hard open problem. Surveys and results on the sizes of plane complete arcs can be found, e.g. in [1–5] and references therein. The exact values of $t_2(2, q)$ are known only for $q \leq 32$, see [5].

This work is devoted to *upper bounds* on $t_2(2,q)$.

Let $t(\mathcal{P}_q)$ be the size of the smallest complete arc in any (not necessarily Galois) projective plane $\mathcal{P}_q$ of order q. In [4], for *sufficiently large* q, the following result is proved by *probabilistic methods* (we give it in the form of [3]):

$$t(\mathcal{P}_q) \le D\sqrt{q}\log^C q,\ C \le 300, \tag{1}$$

where C and D are constants independent of q (i.e. so-called universal or absolute constants). The logarithm basis is not noted as the estimate is asymptotic. The authors of [4] conjecture that the constant can be reduced to $C = 10$ but it has not proved.

Denote by $\bar{t}_2(2,q)$ the smallest *known* size of a complete arc in $PG(2,q)$.

For $q \le 5107$ the values of $\bar{t}_2(2,q)$ (up to August 2011) are collected in [1, Tabs 1-4] whence it follows that:

$$\begin{aligned} \bar{t}_2(2,q) &< 4\sqrt{q} \text{ for } q \le 841,\ q = 31^2, 2^{10}, 37^2, 41^2, 7^4; \\ t_2(2,q) &< 4.5\sqrt{q} \quad \text{for } q \le 2593,\ q = 2693, 2753; \\ t_2(2,q) &< 4.79\sqrt{q} \ \text{ for } q \le 5107; \\ t_2(2,q) &< 0.9987\sqrt{q}\ln^{0.75} q \quad \text{for } 23 \le q \le 5107. \end{aligned} \tag{2}$$

In this work, we obtained more than 800 new small complete arcs in $PG(2,q)$ and extended and improved results of [1], in particular ones of (2). We obtained complete arcs with sizes smaller than in [1], i.e. we *improved upper bounds on* $t_2(2,q)$ *of* [1], for more than 215 values of q with $q \le 5107$. Also, we obtained new arcs for *all* q in the region $5107 < q \le 10111$. We made Observation 1 and we proved Theorem 2 where $G = \{11003, 11971, 11981, 11987, 12007, 13001, 14009, 15013, 16001, 2^{14}, 17011, 18013, 19001, 20011, 21001, 22003, 23011, 24001, 25013, 26003, 27011, 28001, 29009, 30011, 31013, 32003, 2^{15}, 33013, 34019, 35023, 36007, 37003, 38011, 39019, 40009, 41011, 42013, 43003, 44017, 2^{18}\}$.

Observation 1. *In the region* $8000 < q \le 10111$ *and* $q \in G$, *when* q *grows, the positive difference* $\sqrt{q}\ln^{0.73} q - \bar{t}_2(2,q)$ *has a tendency to increase.*

Theorem 2. *In* $PG(2,q)$, *the following holds.*

$$\begin{aligned} t_2(2,q) &< \sqrt{q}\ln^{0.73} q \quad \textit{for } 109 \le q \le 10111,\ q \in G. \\ t_2(2,q) &< 3.8\sqrt{q} \quad \textit{for } q \le 541,\ q = 601, 661. \\ \bar{t}_2(2,q) &< 4\sqrt{q} \quad \textit{for } q \le 841,\ \ q = 857, 31^2, 2^{10}, 37^2, 41^2, 7^4. \\ t_2(2,q) &< 4.3\sqrt{q} \quad \textit{for } q \le 1627,\ q = 1657, 1663, 41^2, 1697, 7^4. \\ t_2(2,q) &< 4.5\sqrt{q} \quad \textit{for } q \le 2647,\ q = 2659, 2663, 2683, 2693, 2753, 2801. \\ t_2(2,q) &< 4.8\sqrt{q} \quad \textit{for } q \le 5419,\ q = 5441, 5443, 5471, 5483, 5501, 5521. \\ t_2(2,q) &< 5\sqrt{q} \quad \textit{for } q \le 9497,\ q = 9539, 9587, 9613, 9649, 9689, 9973. \\ t_2(2,q) &< 5.04\sqrt{q} \ \textit{for } q \le 10111. \\ t_2(2,q) &< 5.5\sqrt{q} \quad \textit{for } q \le 10111,\ q \in G \textit{ with } q \le 40009. \end{aligned} \tag{3}$$

Our results and observations allow us to conjecture the following.

Conjecture 3. *In $PG(2,q)$ it holds that*

$$\begin{array}{lll} t_2(2,q) & < & \sqrt{q}\ln^{0.73} q \quad \textit{for all } q \geq 109. \\ t_2(2,q) & < & 5.5\sqrt{q} \quad\quad \textit{for } q \leq 40009. \end{array}$$

Some results of this work can be found online in [2].

New sizes of small complete arcs obtained in this work are given in Tables 1-5. As far as it is known to the authors, these new sizes are the values of $\bar{t}_2(2,q)$. In other words, the new sizes of small complete arcs obtained in this work are *new upper bounds* on the smallest sizes $t_2(2,q)$ of complete arcs.

In all tables we denote $\bar{t}_2 = \bar{t}_2(2,q)$.

The graphs of values of $\sqrt{q}\ln^{0.73} q$ and $\bar{t}_2(2,q)$ are shown on Fig. 1. For $q < 8000$ the curves practically coalesce with each other, but always $\bar{t}_2(2,q) < \sqrt{q}\ln^{0.73} q$. Moreover, when $q > 8000$ grows, the graphs $\sqrt{q}\ln^{0.73} q$ and $\bar{t}_2(2,q)$ diverge so that the positive difference $\sqrt{q}\ln^{0.73} q - \bar{t}_2(2,q)$ increases.

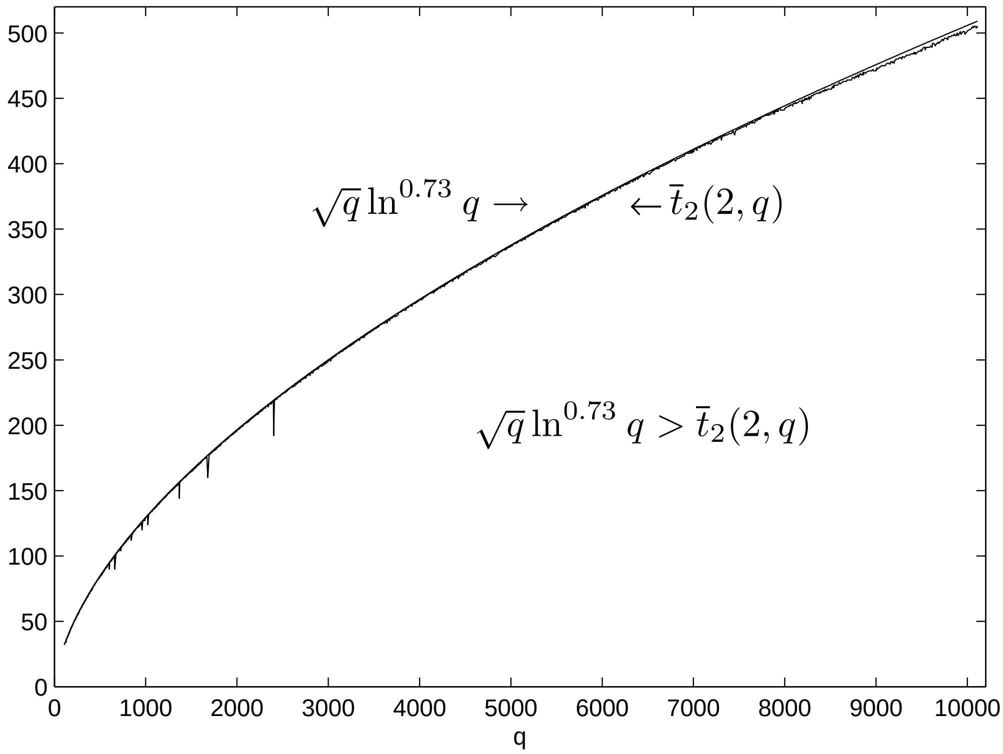

Figure 1: Values of $\sqrt{q}\ln^{0.73} q$ (the top curve) and $\bar{t}_2(2,q)$ (the bottom curve)

Table 1: New sizes $\bar{t}_2 = \bar{t}_2(2,q)$ of complete arcs in $PG(2,q)$,
$\bar{t}_2 < 4\sqrt{q}$ for $q \leq 857$, $4\sqrt{q} < \bar{t}_2 < 4.5\sqrt{q}$ for $881 \leq q \leq 2801$

q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$
359	68	401	73	512	85	571	91	653	99	773	110	857	117
367	69	419	75	541	88	643	98	719	105	787	111		
881	119	1187	143	1511	165	1741	180	2039	198	2269	211	2531	225
919	122	1201	144	1523	166	1759	181	2063	199	2273	211	2609	229
929	123	1217	145	1553	168	1777	182	2111	202	2287	212	2617	230
941	124	1231	146	1567	169	1789	183	2113	202	2309	213	2621	230
953	125	1259	148	1571	169	1823	185	2129	203	2339	215	2633	230
967	126	1289	150	1583	170	1871	188	2131	203	2341	214	2647	231
1019	130	1301	151	1597	171	1873	188	2141	204	2347	215	2659	232
1031	131	1319	152	1601	171	1889	189	2143	204	2357	216	2663	232
1069	134	1331	153	1613	172	1907	190	2179	206	2399	218	2683	233
1097	136	1361	155	1627	173	1973	194	2197	207	2411	219	2801	238
1109	137	1373	156	1663	175	1987	195	2213	208	2417	219		
1123	138	1433	160	1693	177	1993	195	2221	208	2437	220		
1151	140	1447	161	1697	177	2003	196	2237	209	2467	222		
1163	141	1493	164	1723	179	2027	197	2251	210	2473	222		

Table 2: New sizes $\bar{t}_2 = \bar{t}_2(2,q)$ of complete arcs in $PG(2,q)$,
$4.5\sqrt{q} < \bar{t}_2 < 4.8\sqrt{q}$

q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$
2657	232	3019	250	3527	274	3851	288	4507	317	5153	342	5347	350
2677	233	3023	249	3541	275	3877	290	4513	317	5167	344	5351	350
2699	234	3037	251	3547	275	3917	291	4603	321	5171	344	5381	352
2719	235	3041	251	3557	276	3923	292	4621	321	5179	344	5387	352
2741	236	3061	252	3581	277	3943	293	4673	323	5189	344	5393	352
2797	239	3137	256	3613	278	3947	293	4729	325	5197	345	5399	352
2819	240	3181	258	3631	279	3989	295	4751	327	5209	345	5407	352
2833	240	3217	260	3671	281	4007	296	4793	329	5227	346	5413	353
2837	241	3221	260	3673	280	4051	298	4799	329	5231	346	5417	353
2851	242	3259	262	3677	281	4079	299	4903	333	5233	346	5419	353
2857	242	3271	262	3691	282	4096	300	4931	334	5237	346	5441	354
2879	243	3307	264	3697	282	4127	301	4973	336	5261	347	5443	354
2897	244	3329	265	3701	282	4129	301	4999	337	5273	348	5471	355
2917	245	3331	265	3719	283	4201	304	5023	338	5279	348	5483	355
2953	246	3371	267	3721	283	4337	310	5051	339	5281	348	5501	356
2957	247	3373	267	3761	284	4339	310	5077	340	5297	349	5521	356
2971	247	3391	267	3767	285	4391	312	5081	340	5309	349		
2999	249	3407	269	3823	288	4409	313	5099	341	5323	349		
3001	249	3449	271	3833	288	4451	315	5101	341	5329	350		
3011	250	3461	271	3847	288	4483	316	5113	341	5333	349		

Table 3: New sizes $\bar{t}_2 = \bar{t}_2(2,q)$ of complete arcs in $PG(2,q)$, $4.8\sqrt{q} < \bar{t}_2 < 5\sqrt{q}$

q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$
5431	354	6053	377	6619	396	7207	417	7793	437	8419	456	9007	473
5437	354	6067	378	6637	398	7211	418	7817	437	8423	456	9011	472
5449	355	6073	377	6653	398	7213	417	7823	438	8429	455	9013	473
5477	356	6079	378	6659	398	7219	417	7829	437	8431	456	9029	473
5479	356	6089	378	6661	398	7229	418	7841	438	8443	457	9041	474
5503	357	6091	379	6673	399	7237	418	7853	438	8447	456	9043	474
5507	357	6101	379	6679	399	7243	419	7867	440	8461	456	9049	474
5519	357	6113	379	6689	399	7247	419	7873	440	8467	457	9059	475
5527	357	6121	380	6691	399	7253	418	7877	438	8501	457	9067	475
5531	358	6131	380	6701	400	7283	420	7879	440	8513	457	9091	475
5557	358	6133	380	6703	400	7297	421	7883	438	8521	459	9103	475
5563	359	6143	379	6709	400	7307	418	7901	441	8527	458	9109	476
5569	359	6151	380	6719	400	7309	420	7907	440	8537	460	9127	477
5573	359	6163	380	6733	401	7321	421	7919	441	8539	458	9133	477
5581	359	6173	382	6737	401	7331	422	7921	440	8543	460	9137	477
5591	359	6197	382	6761	402	7333	421	7927	440	8563	460	9151	476
5623	361	6199	381	6763	402	7349	422	7933	442	8573	460	9157	477
5639	362	6203	383	6779	403	7351	422	7937	442	8581	460	9161	478
5641	362	6211	383	6781	403	7369	422	7949	442	8597	462	9173	478
5647	362	6217	383	6791	403	7393	423	7951	441	8599	462	9181	477
5651	362	6221	383	6793	403	7411	423	7963	441	8609	461	9187	478
5653	362	6229	383	6803	402	7417	423	7993	441	8623	462	9199	477
5657	363	6241	384	6823	404	7433	425	8009	443	8627	462	9203	479
5659	363	6247	384	6827	404	7451	422	8011	443	8629	462	9209	479
5669	363	6257	384	6829	404	7457	424	8017	443	8641	463	9221	480
5683	363	6263	385	6833	404	7459	425	8039	444	8647	463	9227	480
5689	363	6269	384	6841	404	7477	426	8053	444	8663	463	9239	480
5693	363	6271	384	6857	405	7481	426	8059	445	8669	463	9241	480
5701	364	6277	385	6859	405	7487	426	8069	445	8677	463	9257	481
5711	363	6287	385	6863	405	7489	426	8081	445	8681	464	9277	481
5717	364	6299	385	6869	405	7499	427	8087	444	8689	464	9281	480
5737	365	6301	386	6871	405	7507	427	8089	445	8693	464	9283	481
5741	366	6311	386	6883	406	7517	427	8093	446	8699	464	9293	482
5743	365	6317	387	6889	406	7523	428	8101	446	8707	464	9311	481
5749	365	6323	387	6899	406	7529	428	8111	446	8713	464	9319	482

Table 3: continue

q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$
5779	367	6329	387	6907	407	7537	428	8117	447	8719	464	9323	482
5783	366	6337	386	6911	408	7541	428	8123	446	8731	464	9337	483
5791	367	6343	388	6917	408	7547	428	8147	447	8737	466	9341	483
5801	367	6353	388	6947	408	7549	428	8161	448	8741	466	9343	483
5807	368	6359	387	6949	408	7559	429	8167	448	8747	466	9349	483
5813	366	6361	388	6959	409	7561	429	8171	448	8753	466	9371	483
5821	369	6367	388	6961	408	7573	429	8179	448	8761	465	9377	484
5827	369	6373	389	6967	409	7577	428	8191	446	8779	467	9391	484
5839	369	6379	389	6971	409	7583	429	8192	449	8783	466	9397	484
5843	368	6389	388	6977	410	7589	430	8209	448	8803	468	9403	484
5849	369	6397	389	6983	408	7591	430	8219	450	8807	466	9409	484
5851	370	6421	390	6991	410	7603	430	8221	449	8819	468	9413	485
5857	370	6427	391	6997	409	7607	430	8231	450	8821	468	9419	484
5861	370	6449	391	7001	409	7621	431	8233	449	8831	469	9421	485
5867	370	6451	390	7013	411	7639	432	8237	450	8837	468	9431	485
5869	370	6469	391	7019	411	7643	431	8243	449	8839	468	9433	484
5879	371	6473	392	7027	412	7649	432	8263	451	8849	468	9437	484
5881	371	6481	392	7039	410	7669	431	8269	449	8861	470	9439	484
5897	371	6491	393	7043	412	7673	432	8273	451	8863	468	9461	486
5903	372	6521	392	7057	413	7681	433	8287	450	8867	470	9463	486
5923	372	6529	393	7069	412	7687	433	8291	452	8887	469	9467	486
5927	372	6547	394	7079	413	7691	433	8293	452	8893	470	9473	486
5939	373	6551	395	7103	414	7699	433	8297	452	8923	471	9479	486
5953	372	6553	395	7109	414	7703	434	8311	452	8929	471	9491	487
5981	375	6561	395	7121	414	7717	434	8317	453	8933	471	9497	486
5987	374	6563	395	7127	414	7723	435	8329	453	8941	471	9539	487
6007	375	6569	395	7129	415	7727	434	8353	454	8951	471	9587	489
6011	376	6571	396	7151	416	7741	435	8363	454	8963	472	9613	489
6029	375	6577	396	7159	415	7753	436	8369	454	8969	472	9649	490
6037	377	6581	396	7177	416	7757	436	8377	454	8971	471	9689	492
6043	377	6599	396	7187	415	7759	436	8387	455	8999	471	9973	499
6047	377	6607	397	7193	415	7789	437	8389	454	9001	472		

Table 4: New sizes $\bar{t}_2 = \bar{t}_2(2,q)$ of complete arcs in $PG(2,q)$, $5\sqrt{q} < \bar{t}_2 < 5.04\sqrt{q}$

q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$
9511	488	9631	492	9739	494	9811	497	9883	498	9967	501	10091	505
9521	489	9643	491	9743	495	9817	497	9887	500	10007	502	10093	504
9533	489	9661	492	9749	496	9829	498	9901	498	10009	502	10099	505
9547	489	9677	492	9767	496	9833	498	9907	499	10037	503	10103	504
9551	489	9679	493	9769	496	9839	498	9923	501	10039	503	10111	505
9601	490	9697	493	9781	495	9851	498	9929	499	10061	505		
9619	492	9719	494	9787	495	9857	499	9931	499	10067	504		
9623	492	9721	495	9791	497	9859	499	9941	500	10069	505		
9629	492	9733	495	9803	497	9871	498	9949	501	10079	505		

Table 5: New sizes $\bar{t}_2 = \bar{t}_2(2,q)$ of complete arcs in $PG(2,q)$ for big q, $\overline{\Phi}_q = \sqrt{q}\ln^{0.73} q - \bar{t}_2$

q	$\bar{t}_2$	$\overline{\Phi}_q$	q	$\bar{t}_2$	$\overline{\Phi}_q$	q	$\bar{t}_2$	$\overline{\Phi}_q$	q	$\bar{t}_2$	$\overline{\Phi}_q$
11003	530	4.5	17011	679	8.2	27011	881	14.7	36007	1040	15.4
11971	557	4.2	18013	702	8.2	28001	901	13.3	37003	1055	16.9
11981	557	4.5	19001	723	9.3	29009	918	15.0	38011	1070	18.4
11987	556	5.6	20011	747	7.4	30011	934	17.3	39019	1085	19.8
12007	558	4.2	21001	765	10.5	31013	955	14.3	40009	1098	22.6
13001	583	5.6	22003	785	11.5	32003	969	17.8	41011	1118	18.5
14009	609	5.5	23011	805	12.2	2^{15}	990	10.2	42013	1131	21.2
15013	633	6.5	24001	827	10.2	33013	988	16.4	43003	1145	22.6
16001	656	7.4	25013	843	14.2	34019	1004	17.8	44017	1163	20.1
2^{14}	665	7.5	26003	864	12.5	35023	1019	19.8	2^{18}	3066	165.6

References

[1] D. Bartoli, A. A. Davydov, G. Faina, S. Marcugini, and F. Pambianco, On sizes of complete arcs in $PG(2,q)$, *Discrete Math.* **312**, 680–698, 2012.

[2] D. Bartoli, A. A. Davydov, G. Faina, S. Marcugini, and F. Pambianco, Upper bounds on the smallest size of a complete arc in the plane $PG(2,q)$, http://arxiv.org/abs/1111.3403

[3] J. W. P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces, *J. Statist. Planning Infer.* **72**, 355–380, 1998.

[4] J. H. Kim and V. Vu, Small complete arcs in projective planes, *Combinatorica* **23**, 311–363, 2003.

[5] S. Marcugini, A. Milani, and F. Pambianco, Minimal complete arcs in $PG(2,q)$, $q \leq 32$, in *Proc. XII Int. Workshop on Algebraic and Combin. Coding Theory, ACCT2010, Novosibirsk, Russia, 2010,* 217–222.