Are q-ary Perfect Codes reconstructed by the Vertices of Largest Level?

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ACCT-XIV

Notations

$$\mathbf{F}_q = \{0, 1, \ldots, q-1\}, \quad \mathbf{F}_q^n = \mathbf{F}_q \times \ldots \times \mathbf{F}_q$$

$$s(\alpha) = \{i \ : \alpha_i \neq 0\}, \quad \alpha \in \mathbf{F}_q^n$$

$$wt(\alpha) = |s(\alpha)|, \qquad \rho(\alpha, \beta) = wt(\beta - \alpha)$$

$$W_i = \{\beta \in \mathbf{F}_q^n \ : \ wt(\beta) = i\}$$

$$B_i = W_0 \cup W_1 \cup \ldots \cup W_i$$

Eigenfunctions: definition

$$V = \{f : F_q^n \longrightarrow \mathbb{C}\}$$

$$f \leftrightarrow (f(0,\dots,0), f(0,\dots,0,1),\dots, f(q-1,\dots,q-1))^T$$

$$D - q^n \times q^n, \quad D_{\alpha,\beta} = \begin{cases} 1, & \rho(\alpha,\beta) = 1 \\ 0, & other \end{cases}$$ – adjacency matrix of $\mathbf{F}_q^n$.

Eigenfunction

f is the eigenfunction of $\mathbf{F}_q^n$ with the eigenvalue λ (or λ-function) if

$$Df = \lambda f.$$

In other words, $\sum_{\beta \in W_1(\alpha)} f(\beta) = \lambda f(\alpha), \quad \forall \alpha \in \mathbf{F}_q^n.$

The eigenvalues of the graph of n-dimensional q-ary hypercube are equal to $\lambda_i = (q-1)n - qi, \; i = 0,1,\dots,n$

Perfect codes: definition

Definition

The code $C \subseteq \mathbf{F}_q^n$ is perfect if

$$\forall \alpha \in \mathbf{F}_q^n \ \ \exists! \ \ \beta \in \mathbf{F}_q^n \ \ \rho(\alpha, \beta) = 1$$

A perfect code intersects with a ball of radius 1 by exactly one vertex.
Let χ_C be the characteristic function of C
then $\chi_C - 1/((q-1)n+1)$ is λ-function with $\lambda = -1$

The number of the eigenvalue for perfect codes

The number of the eigenvalue is $h(-1) = ((q-1)n+1)\,/q$

Reconstruction of Perfect Binary Codes

$n = 2^t - 1$ is odd

$$|W_{(n-1)/2}| = |W_{(n+1)/2}| = max_{i=0,1,\dots,n}|W_i|$$

Reconstruction

Avgustinovich S.V.:

$$C \bigcap W_{(n-1)/2} \rightarrowtail C$$

The idea: Any perfect binary code is antipodal: $\alpha \in C \Leftrightarrow \mathbf{1} + \alpha \in C$

$$C \cap W_{(n-1)/2} \rightarrowtail C \cap \left(W_{(n-1)/2} \cup W_{(n+1)/2}\right)$$

$$C_1 \neq C_2, \quad C_1 \cap W_{(n-1)/2} = C_2 \cap W_{(n-1)/2}$$

$$\left(C_1 \cap \left(W_0 \cup W_1 \cup \ldots \cup W_{(n+1)/2}\right)\right) \cup \left(C_2 \cap \left(W_{(n+1)/2} \cup \ldots \cup W_n\right)\right)$$

λ-functions: binary case

Any λ-function code is antipodal or minus-antipodal: $f(\alpha) = \pm f(\alpha)$

Reconstruction

As in case of perfect codes: $f(\alpha), \alpha \in W_m \rightarrowtail f(\alpha), \alpha \in \mathbf{F}_q^n$,
where $m = (n \pm 1)/2$

V_h denotes the space of all λ-functions with $\lambda = n - 2h$

$$V_h = \langle f^\alpha(\beta) = (-1)^{\alpha\beta} \ : \ wt(\alpha) = h \rangle$$

Krawtchouk polynomials

$$(x-y)^t(x+(q-1)y)^{N-t} = \sum_{m=0}^{N} P_m^{(q)}(t;N)y^m x^{N-m},$$

$$P_m^{(q)}(t;N) = \sum_{j=0}^{m}(-1)^j(q-1)^{m-j}\binom{t}{j}\binom{N-t}{m-j} -$$

Krawtchouk polynomial

λ-functions: binary case

$\lambda, \quad h = h(\lambda) = (n - \lambda)/2, \;\; f$ - λ-function.

Reconstruction

$f(\alpha), \alpha \in W_h \;\rightarrowtail\; f(\alpha), \alpha \in \mathbf{F}_q^n$
in case
$P_h(h; n), \ldots, P_{h-k}(h-k; n-2k), \ldots, P_0(0; n-2h)$ are nonzero.

Reconstruction

Let $d \leq h$. Then
$f(\alpha), \alpha \in W_d \;\rightarrowtail\; f(\alpha), \alpha \in B_d$ if
$P_d(h; n), \ldots, P_{d-k}(h-k; n-2k), \ldots, P_0(h-d; n-2h)$ are nonzero.

λ-functions: q-ary case

Let λ be an eigenvalue of the hypercube $\mathbf{F}_q^n$ and $d \leq l(\lambda)$.
We know the values $f(\alpha)$ for all α with Hamming weight d.
Question: Is it possible to determine (uniquely) the values $f(\alpha)$ for all α with Hamming weight less than d?
Obviously,

$$\sum_{\alpha \in W_d} f(\alpha) = P_d^{(q)}(h;n) f(\mathbf{0}).$$

Then

$$f(\mathbf{0}) = \frac{\sum_{\alpha \in W_d} f(\alpha)}{P_d^{(q)}(h;n)}$$

if $P_d^{(q)}(h;n) \neq 0$.

λ-functions: q-ary case

$f(\alpha), \alpha \in W_d$ – are known

λ-functions: q-ary case

$f(\alpha), \alpha \in W_d$ – are known
$f(\alpha), \alpha \in W_0$ – can be calculated

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$f(\alpha), \alpha \in W_d$ – are known
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$f(\alpha), \alpha \in W_1 \bigcup \ldots \bigcup W_{k-1},\ k < d$ – assume that can be calculated

λ-functions: q-ary case

$f(\alpha), \alpha \in W_d$ – are known
$f(\alpha), \alpha \in W_0$ – can be calculated
$f(\alpha), \alpha \in W_1 \bigcup \ldots \bigcup W_{k-1},\ k < d$ – assume that can be calculated
$f(\alpha), \alpha \in W_k$ – let us calculate!

λ-functions: q-ary case

Theorem

Let λ be an eigenvalue of $\mathbf{F}_q^n$, $n-h<h$, $d \le h$ and $\varphi : W_d \longrightarrow C$ be a function. Suppose that f is a λ-function such that for any $\alpha \in W_d$ it holds $f(\alpha)=\varphi(\alpha)$. Then for any $\alpha \in B_d$ the value $f(\alpha)$ is uniquely determined if for all $k=0,\ldots,d$ and $l=0,\ldots,k$

$$\sum_{i=0}^{k} r^k_{i,d-k} P_i^{(q-1)}(l,k) \neq 0, \qquad (1)$$

where r^k_{ij} is defined in terms of Krawtchouk polynomials.

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Thank you for your attention!