New 4-dimensional linear codes over $\mathbb{F}_9$

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1. Optimal linear codes problem

$\mathbb{F}_q$: the field of q elements

$\mathbb{F}_q^n = \{(a_1, \cdots, a_n) \mid a_i \in \mathbb{F}_q\}$

The weight of $\boldsymbol{a} = (a_1, \cdots, a_n) \in \mathbb{F}_q^n$ is

$$wt(\boldsymbol{a}) = |\{i \mid a_i \neq 0\}|$$

An $[n,k,d]_q$ code $\mathcal{C}$ means a k-dimensional subspace of $\mathbb{F}_q^n$ with minimum weight d,

$$d = \min\{wt(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathcal{C}, \boldsymbol{a} \neq \boldsymbol{0}\}.$$

A vector $\boldsymbol{a} \in \mathcal{C}$ is called a codeword.

For an $[n,k,d]_q$ code $\mathcal{C}$, a $k \times n$ matrix G whose rows form a basis of $\mathcal{C}$ is a generator matrix.

The weight distribution (w.d.) of $\mathcal{C}$ is the list of numbers $A_i > 0$, where

$$A_i = |\{\boldsymbol{c} \in \mathcal{C} \mid wt(\boldsymbol{c}) = i\}| > 0.$$

The weight distribution

$$(A_0, A_d, ...) = (1, \alpha, ...)$$

is also expressed as

$$0^1 d^\alpha \cdots .$$

A good $[n, k, d]_q$ code will have

small n for fast transmission of messages,

large k to enable transmission of a wide variety of messages, and

large d to correct many errors.

The problem to optimize one of the parameters n, k, d for given the other two is called "optimal linear codes problem" (Hill 1992).

Problem 1. Find $n_q(k,d)$, the smallest value of n for which an $[n,k,d]_q$ code exists.

Problem 2. Find $d_q(n,k)$, the largest value of d for which an $[n,k,d]_q$ code exists.

An $[n,k,d]_q$ code is called optimal if

$$n = n_q(k,d) \text{ or } d = d_q(n,k).$$

We deal with Problem 1 for $q = 9$, $k = 4$.

The Griesmer bound

$$n \geq g_q(k,d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil$$

where $\lceil x \rceil$ is a smallest integer $\geq x$.

An $[n,k,d]_q$ code attaining the Griesmer bound is called a Griesmer code.
Griesmer codes are optimal.

Known results for $q = 9$, $k = 4$

The exact values of $n_9(4, d)$ are determined for all d for $d \geq 1216$.

For $1 \leq d \leq 1215$, $n_9(4, d)$ is detemined for 276 values of d but not for 939 values of d.

2. The geometric method

PG(r, q): projective space of dim. r over $\mathbb{F}_q$

j-flat: j-dim. projective subspace of PG(r, q)

0-flat: point   1-flat: line

2-flat: plane $(r-1)$-flat: hyperplane

$\theta_j := (q^{j+1}-1)/(q-1) = q^j + q^{j-1} + \cdots + q + 1$

$\mathcal{C}$: an $[n,k,d]_q$ code generated by G.

Since we would like to find $n_q(k,d)$,

we assume that G contains no all-zero-columns.

Then the columns of G can be considered

as a multiset of n points in $\Sigma = \mathrm{PG}(k-1,q)$

denoted also by $\mathcal{C}$.

$\mathcal{F}_j :=$ the set of j-flats of Σ

i-point: a point of Σ with multiplicity i in $\mathcal{C}$.

γ_0: the maximum multiplicity of a point from Σ in $\mathcal{C}$

C_i: the set of i-points in Σ, $0 \leq i \leq \gamma_0$.

$\lambda_i := |C_i|$, $0 \leq i \leq \gamma_0$.

For $^\forall S \subset \Sigma$, the multiplicity of S w.r.t. $\mathcal{C}$, denoted by $m_{\mathcal{C}}(S)$, is defined by

$$m_{\mathcal{C}}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|.$$

Then we obtain the partition

$\Sigma = C_0 \cup C_1 \cup \cdots \cup C_{\gamma_0}$ such that

$$
\begin{aligned}
n &= m_{\mathcal{C}}(\Sigma), \\
n - d &= \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\}.
\end{aligned}
$$

Conversely such a partition of Σ as above gives an $[n, k, d]_q$ code in the natural manner.

i-hyperplane: a hyperplane π with $i = m_{\mathcal{C}}(\pi)$.

$a_i := |\{\pi \in \mathcal{F}_{k-2} \mid m_{\mathcal{C}}(\pi) = i\}|$.

The list of a_i's is the spectrum of $\mathcal{C}$.

$a_i = A_{n-i}/(q-1)$ for $0 \leq i \leq n-d$.

3. Projective dual

An $[n, k, d]_q$ code is m-divisible (or m-div) if

$\exists m > 1$ s.t. $A_i > 0 \Rightarrow m|i$.

Ex. 1. There exists a 3-div $[41, 4, 33]_9$ code with w.d. $0^1 33^{984} 36^{3608} 39^{1968}$.

The spectrum is $(a_2, a_5, a_8) = (246, 451, 123)$.

Lemma 1. (Projective dual)

$\mathcal{C}$: m-div $[n,k,d]_q$ code, $q = p^h$, p prime.

$m = p^r$ for some $1 \leq r < h(k-2)$, $\lambda_0 > 0$.

$\Rightarrow \exists \mathcal{C}^*$: t-div $[n^*,k,d^*]_q$ code with

$$t = q^{k-2}/m,$$

$$n^* = ntq - \frac{d}{m}\theta_{k-1},$$

$$d^* = ((n-d)q - n)t.$$

A generator matrix for $\mathcal{C}^*$ is given by considering $(n-d-jm)$-hyperplanes as j-points in the dual space Σ^* of Σ for $0 \leq j \leq w-1$.

Ex. 2.

$\mathcal{C}$: 3-div $[41, 4, 33]_9$

with spec. $(a_2, a_5, a_8) = (246, 451, 123)$

$\downarrow$ projective dual

$\mathcal{C}^*$: 27-div $[943, 4, 837]_9$ $(n^* = 2a_2 + a_5)$

with spec. $(a^*_{79}, a^*_{106}) = (41, 779)$

4. Geometric puncturing

The puncturing from a given $[n, k, d]_q$ code by deleting the coordinates corresponding to some geometric object in $\Sigma = \mathrm{PG}(k-1, q)$ is geometric puncturing.

Lemma 2. $\mathcal{C}$: $[n, k, d]_q$ code
$\cup_{i=0}^{\gamma_0} C_i$: the partition of Σ obtained from $\mathcal{C}$.
If $\cup_{i \geq 1} C_i$ contains a t-flat Π and if $d > q^t$
$\Rightarrow \exists \mathcal{C}'$: $[n - \theta_t, k, d']_q$ code, for $d' \geq d - q^t$.

5. Quasi-cyclic codes

$R = \mathbb{F}_q[x]/(x^N - 1)$: ring of polynomials over $\mathbb{F}_q$ modulo $x^N - 1$.

We associate $(a_0, a_1, ..., a_{N-1}) \in \mathbb{F}_q^N$ with $a_0 + a_1 x + \cdots + a_{N-1} x^{N-1} \in R$.

For $\mathbf{g} = (g_1(x), \cdots, g_m(x)) \in R^m$, an ideal $C_\mathbf{g}$ of R^m defined by

$$C_\mathbf{g} = \{(r(x)g_1(x), \cdots, r(x)g_m(x)) \mid r(x) \in R\}$$

is called the 1-generator quasi-cyclic (QC) code with generator $\mathbf{g}$.

When $m = 1$, $\mathcal{C} = C_{\mathbf{g}}$ is called cyclic satisfying that $c(x) \in \mathcal{C}$ implies $x \cdot c(x) \in \mathcal{C}$,

i.e., $(c_0, c_1, ..., c_{N-1}) \in \mathcal{C}$

$\Rightarrow \quad (c_{N-1}, c_0, c_1, ..., c_{N-2}) \in \mathcal{C}$.

Let $g(x) = x^k - \sum_{i=0}^{k-1} g_i x^i \in \mathbb{F}_q[x]$ dividing $x^N - 1$.

We denote by $[g^N]$ or by $[g_0 g_1 \cdots g_{k-1}^N]$ the $k \times N$ matrix

$$[P, TP, T^2P, ..., T^{N-1}P],$$

where

$$T = \left[\begin{array}{ccccc|c} 0 & 0 & \dots & \dots & 0 & g_0 \\ \hline 1 & 0 & \dots & \dots & 0 & g_1 \\ 0 & 1 & 0 & \dots & 0 & g_2 \\ 0 & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & \dots & 0 & \ddots & 0 & g_{k-2} \\ 0 & \dots & \dots & 0 & 1 & g_{k-1} \end{array}\right], P = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

i.e., T is the companion matrix of $g(x)$.

$$\tau : \mathsf{PG}(k-1,q) \longrightarrow \mathsf{PG}(k-1,q)$$

defined by

$$\tau(\mathbf{P}(x_0,\cdots,x_{k-1})) = \mathbf{P}(T(x_0,\cdots,x_{k-1})^{\mathsf{T}}).$$

Then the columns of $[g^N]$ can be considered as an orbit of τ.

Now, take m orbits $\mathcal{O}_1,\mathcal{O}_2,\cdots,\mathcal{O}_m$ of τ with length N, and select a point P_i from each $\mathcal{O}_i$. We take $P_1,P_2,\cdots,P_m$ as non-zero column vectors in $\mathbb{F}_q^k$.

We always take P_1 as $P = (1, 0, 0, \cdots, 0)^{\mathsf{T}}$.

We denote the matrix

$$[P_1, TP_1, T^2P_1, ..., T^{n_1-1}P_1; P_2, TP_2, \cdots$$
$$\cdots ; P_m, TP_m, T^2P_m, ..., T^{n_m-1}P_m]$$

by $[g^{n_1}] + P_2^{n_2} + \cdots + P_m^{n_m}$.

Then, the matrix $[g^N]+P_2^N+\cdots+P_m^N$ defined from m orbits $\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_m$ of τ generates a QC code.

$\mathbb{F}_9 = \{0, 1, \alpha, \cdots, \alpha^7\}$,with $\alpha^2 = \alpha + 1$.

we denote $\alpha, \alpha^2, \cdots, \alpha^7$ by $2, 3, \cdots, 8$ so that

$\mathbb{F}_9 = \{0, 1, 2, \cdots, 8\}$.

addition table

+	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	5	3	8	7	0	4	6	2
2	2	3	6	4	1	8	0	5	7
3	3	8	4	7	5	2	1	0	6
4	4	7	1	5	8	6	3	2	0
5	5	0	8	2	6	1	7	4	3
6	6	4	0	1	3	7	2	8	5
7	7	6	5	0	2	4	8	3	1
8	8	2	7	6	0	3	5	1	4

multiplication table

$\times$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	3	4	5	6	7	8	1
3	0	3	4	5	6	7	8	1	2
4	0	4	5	6	7	8	1	2	3
5	0	5	6	7	8	1	2	3	4
6	0	6	7	8	1	2	3	4	5
7	0	7	8	1	2	3	4	5	6
8	0	8	1	2	3	4	5	6	7

Ex. 3.

$$P_1 = (1,0,0,0)^\mathsf{T}\ ,\ P_2 = (1,0,1,7)^\mathsf{T}\ \in\ \mathbb{F}_9^4$$

$$g(x) = x^4 - 5x^3 - 5x^2 - 5x - 5 \in \mathbb{F}_9[x]$$

$$\Rightarrow T = \begin{bmatrix} 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$P_1 = (1,0,0,0)^\mathsf{T} \quad , \quad TP_1 = (0,1,0,0)^\mathsf{T} \ ,$$

$$T^2P_1 = (0,0,1,0)^\mathsf{T} \ , \quad T^3P_1 = (0,0,0,1)^\mathsf{T},$$

$$T^4P_1 = (5,5,5,5)^\mathsf{T} \ , \quad T^5P_1 = (1,0,0,0)^\mathsf{T}$$

$$\mathcal{O}_1 = \{P_1, TP_1, T^2P_1, T^3P_1, T^4P_1\}$$

$$\Rightarrow \quad [1\ 2\ 3\ 4^5] = \left[\begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

$\Rightarrow$ $[5,4,2]_9$ code

$$P_2 = (1,0,1,7)^\top \quad , \quad TP_2 = (3,8,3,8)^\top \ ,$$

$$T^2P_2 = (4,5,0,5)^\top \ , \quad T^3P_2 = (1,7,0,1)^\top,$$

$$T^4P_2 = (5,0,4,5)^\top \ , \quad T^5P_2 = (1,0,1,7)^\top$$

$$\mathcal{O}_2 = \{P_2, TP_2, T^2P_2, T^3P_2, T^4P_2\}$$

$$\Rightarrow \quad 1\ 0\ 1\ 7^5 = \begin{bmatrix} 1 & 3 & 4 & 1 & 5 \\ 0 & 8 & 5 & 7 & 0 \\ 1 & 3 & 0 & 0 & 4 \\ 7 & 8 & 5 & 1 & 5 \end{bmatrix}$$

$\Rightarrow$ $[5,4,2]_9$ code

$$[1\,2\,3\,4^5] = \begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix},\ 1\,0\,1\,7^5 = \begin{bmatrix} 1 & 3 & 4 & 1 & 5 \\ 0 & 8 & 5 & 7 & 0 \\ 1 & 3 & 0 & 0 & 4 \\ 7 & 8 & 5 & 1 & 5 \end{bmatrix}$$

$$\Rightarrow\ [1\,2\,3\,4^5]{+}1\,0\,1\,7^5 = \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 1 & 1 & 3 & 4 & 1 & 5 \\ 0 & 1 & 0 & 0 & 2 & 0 & 8 & 5 & 7 & 0 \\ 0 & 0 & 1 & 0 & 3 & 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 4 & 7 & 8 & 5 & 1 & 5 \end{array}\right]$$

$\Rightarrow$ QC $[10, 4, 5]_9$ code

6. Construction of new codes

Lemma 3. There exists $[1227, 4, 1089]_9$, $[1237, 4, 1098]_9$, $[1247, 4, 1107]_9$ codes.

Proof.

$\mathcal{C}$: QC code with generator matrix

$[5214^7] + 1402^7 + 1371^7 + 1706^7 + 1243^7 + 1377^7$

$+1718^7$.

Then $\mathcal{C}$ is a 3-div $[49, 4, 39]_9$ code with spectrum $(a_1, a_4, a_7, a_{10}) = (28, 448, 267, 77)$.

As projective dual, we get a $[1247, 4, 1107]_9$ code $\mathcal{C}^*$ with w.d. $0^1 1107^{6168} 1134^{392}$.

The multiset for $\mathcal{C}^*$ has two skew lines

$$l_1 = \langle 1000, 1111 \rangle,\ l_2 = \langle 1002, 1121 \rangle.$$

Hence, we get

$[1227, 4, 1089]_9$, $[1237, 4, 1098]_9$ codes by geometric puncturing. □

Lemma 4. There exist $[913, 4, 810]_9$, $[923, 4, 819]_9$, $[933, 4, 828]_9$ and $[943, 4, 837]_9$ codes.

Proof.

$\mathcal{C}$: extended QC code with generator matrix

$[1000^4] + 7211^4 + 1116^4 + 1574^4 + 1376^4 + 1507^4$

$+1247^4 + 1426^4 + 1237^4 + 1860^4 + 1515^1$.

$\Rightarrow$ $\mathcal{C}$: 3-div $[41, 4, 33]_9$ code with spectrum

$$(a_2, a_5, a_8) = (246, 451, 123).$$

$\mathcal{C}$: 3-div $[41, 4, 33]_9$

$\downarrow$ projective dual

$\mathcal{C}^*$: 27-div $[943, 4, 837]_9$

The multiset for $\mathcal{C}^*$ contains three skew lines

$l_1 = \langle 1000, 1018 \rangle$, $l_2 = \langle 1002, 1102 \rangle, l_3 = \langle 1003, 1114 \rangle$.

Hence, we get

$[913, 4, 810]_9$, $[923, 4, 819]_9$ and $[933, 4, 828]_9$

codes by geometric puncturing. $\square$

There are 200 orbits of length 4,

8 orbits of length 2 and 4 fixed points

under the projectivity defined by the companion matrix of $x^4 - 1$.

We give three other 3-divisible codes constructed from these orbits.

Lemma 5. There exist

$[1034 - 10t, 4, 918 - 9t]_9$ codes

for $t = 0, 1, 2, 3, 4, 5, 6, 7, 8.$

Proof.

$\mathcal{C}$: $[38, 4, 30]_9$ code with generator matrix

$$G = [1000^4] + 1721^4 + 1215^4 + 1056^4 + 1574^4$$

$$+1542^4 + 1761^4 + 1065^4 + 1168^4 + 1515^1 + 1357^1,$$

where the columns of G consist of nine orbits of length 4 and two fixed points under the projectivity defined by the companion matrix of $x^4 - 1$.

$\Rightarrow$ $\mathcal{C}$: 3-div $[38, 4, 30]_9$ code with spectrum

$$(a_2, a_5, a_8) = (298, 438, 84).$$

$\mathcal{C}$: 3-div $[38, 4, 30]_9$

$\downarrow$ projective dual

$\mathcal{C}^*$: 27-div $[1034, 4, 918]_9$

The multiset for $\mathcal{C}^*$ contains eight skew lines

$\langle 1000, 1103\rangle$, $\langle 1002, 1111\rangle$, $\langle 1003, 1017\rangle$, $\langle 1005, 1121\rangle$,

$\langle 1006, 1132\rangle$, $\langle 1007, 1140\rangle$, $\langle 1008, 1150\rangle$, $\langle 1010, 1105\rangle$.

Hence, we get

$[1034 - 10t, 4, 918 - 9t]_9$ codes for $1 \leq t \leq 8$

by geometric puncturing. $\square$

Lemma 6. There exist

$$[1125-10t, 4, 999-9t]_9 \text{ codes}$$

$$\text{for } t = 0, 1, 2, 3, 4, 5, 6, 7, 8.$$

Proof.

$\mathcal{C}$: $[35, 4, 27]_9$ code with generator matrix

$$G = 1018^4 + 1077^4 + 1220^4 + 1550^4 + 1034^4 + 1566^4$$

$$+1356^4 + 1313^2 + 1652^2 + 1357^1 + 1111^1 + 1753^1,$$

where the columns of G consist of
seven orbits of length 4,
two orbits of length 2 and three fixed points
under the projectivity defined by the companion matrix of $x^4 - 1$.

$\Rightarrow$ $\mathcal{C}$: 3-div $[35, 4, 27]_9$ code with spectrum

$$(a_2, a_5, a_8) = (360, 405, 55).$$

$\mathcal{C}$: 3-div $[35, 4, 27]_9$

$\downarrow$ projective dual

$\mathcal{C}^*$: 27-div $[1125, 4, 999]_9$

The multiset for $\mathcal{C}^*$ contains eight skew lines

$\langle 1000, 1001 \rangle$, $\langle 1011, 1100 \rangle$, $\langle 1012, 1114 \rangle$, $\langle 1013, 1120 \rangle$,

$\langle 1014, 1130 \rangle$, $\langle 1015, 1140 \rangle$, $\langle 1016, 1150 \rangle$, $\langle 1017, 1161 \rangle$.

Hence, we get

$[1125 - 10t, 4, 999 - 9t]_9$ codes for $1 \le t \le 8$

by geometric puncturing. $\square$

Lemma 7. There exist

$[1186-10t, 4, 1053-9t]_9$ for $t = 0, 1, 2, 3, 4, 5$

and $[1277, 4, 1134]_9$ codes.

Proof.

$\mathcal{C}$: $[39, 4, 30]_9$ code with generator matrix

$$G = [1000^4] + 1721^4 + 1846^4 + 1473^4 + 1300^4 +$$

$$1851^4 + 1574^4 + 1281^4 + 1405^4 + 1256^2 + 1515^1,$$

where the columns of G consist of
nine orbits of length 4,
one orbit of length 2 and one fixed point
under the projectivity defined by the companion matrix of $x^4 - 1$.

$\Rightarrow$ $\mathcal{C}$: 3-div $[39, 4, 30]_9$ code with spectrum

$$(a_0, a_3, a_6, a_9) = (32, 427, 327, 34).$$

$\mathcal{C}$: 3-div $[39, 4, 30]_9$

$\downarrow$ projective dual

$\mathcal{C}^*$: 27-div $[1277, 4, 1134]_9$

The multiset for $\mathcal{C}^*$ contains one plane

$$\langle 1004, 1018, 1118 \rangle$$

and five skew lines

$\langle 1000, 1015 \rangle$, $\langle 1002, 1103 \rangle$, $\langle 1003, 1110 \rangle$, $\langle 1005, 1120 \rangle$, $\langle 1006, 1140 \rangle$.

Hence, we get

$[1186 - 10t, 4, 1053 - 9t]_9$ codes for $0 \leq t \leq 5$ by geometric puncturing. $\square$

7. New results on $n_9(4, d)$

We determined $n_9(4, d)$ for 115 values of d.

(1) $n_9(4, d) = g_9(4, d)$ for $d \in \{811\text{-}837, 892\text{-}918, 973\text{-}999\}$

(2) $n_9(4, d) = g_9(4, d) + 1$ for $d \in \{964\text{-}972, 1045\text{-}1053, 1114\text{-}1116, 1122\text{-}1134\}$

(3) $n_9(4, d) \le g_9(4, d)+1$ for $d \in \{802\text{-}810, 838\text{-}891, 919\text{-}963, 1000\text{-}1044, 1081\text{-}1113, 1117\text{-}1121\}$

Still $n_9(4, d)$ is not determined for 824 values of d.

Thank you for your attention!

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