# Linear coordinates and symmetry groups of Mollard codes

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# Outline

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# Binary codes

A *binary code* of length $n$ is a collection of vectors from $F_2^n$.

# Binary perfect codes

A binary code with the minimum distance 3 is *perfect* if $|C| = 2^n/(n+1)$.

*Remark*: WLOG all codes contain all-zero vector.

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# Hamming code

A linear (over $F_2$) perfect code is called *a binary Hamming code*.

# Steiner triple system

The codewords of weight 3 of a perfect code $C$ form STS, which is denoted by $STS(C)$.

*Remark*: we use a mixed code-design language for Steiner triple systems.

# The symmetry group of a code

For $x \in F_2^n$ and $\pi \in Sym(\{1,\ldots,n\})$ define

$$\pi(x) = (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$$

Given a code $C$ of length $n$, define its *symmetry group* $Sym(C) = \{\pi \in Sym(\{1,\ldots,n\}) : \pi(C) = C\}$.

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# Invariants of perfect code

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# $\mu$-linear coordinates of a perfect code

Let $C$ be a perfect code of length $n$, then denote by

$$\mu_i(C) = |\{x \in \mathrm{STS}(C) \cap \mathrm{Ker}(C) : x_i = 1\}|.$$

$0 \leq \mu_i(C) \leq (n-1)/2$

Define $i$ to be *a $\mu$-linear coordinate* of $C$ if $\mu_i(C) = (n-1)/2$.

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The set of $\mu$-linear coordinates of $C$ is denoted by $Lin_{\mu}(C)$.

## Property

A perfect code $C$ of length $n$ is Hamming iff $Lin_{\mu}(C) = \{1, \ldots, n\}$

# Example

Let $C$ be a $Z_2Z_4$-linear perfect code. Then its $\mu$-linear coordinates are $Z_2$-linear coordinates of $C$.

# Linear coordinates of a perfect code

**Theorem**

The subcode of a perfect code $C$ on the coordinates $Lin_\mu(C)$ is a Hamming code of $C$.

# Mollard code

Let $C$ and $D$ be two codes of lengths $t$ and $m$.

Let $x$ be in $F_2^{tm}$.The coordinates of $x$ are indexed by elements of $\{1,\ldots,t\} \times \{1,\ldots,m\}$.

$$p_1(x) = (\sum_{j=1}^{m} x_{1j}, \ldots, \sum_{j=1}^{m} x_{tj})$$

$$p_2(x) = (\sum_{i=1}^{t} x_{i1}, \ldots, \sum_{i=1}^{t} x_{im})$$

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# The Mollard code

## Mollard code

$M(C,D)=\{(x,p_1(x)+y,p_2(x)+z+f(y)) : x\in F_2^{tm}, y\in C, z\in D\}$. In the talk $f$ is the zero function.

# Mollard code

**Theorem**

If $C$ and $D$ are perfect codes, then $M(C, D)$ is perfect.

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If $S_1$ and $S_2$ are Steiner triple systems (treated as binary codes with all-zero vectors), then $M(S_1, S_2)$ is a STS with all-zero vector.

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# Subcodes of Mollard code

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Describe $Sym(M(C,D))$. $Stab_{D^2}Sym(M(C,D)) = ?$

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# Main results

**Theorem**

Let $C$ and $D$ be two perfect codes. Then

$$Stab_{D^2}(Sym(M(C,D))) \cong$$

$$(Sym(C) \leftthreetimes Z_2^{(log_2(1+|Lin_\mu(C)|))^{t-\mathrm{rank}(C)}}) \times Sym(D).$$

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Given STS $S$ of order $n$ define a point $i$ to be *$\nu$-linear* if $i$ is in $(n-1)(n-3)/4$ Pasch configurations of $S$.

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# Conclusion

- *The characteristic $\mu_i(C)$ and the notion of $\mu$-linear coordinate for a perfect code are suggested*
- *The description for $Stab_{D^2}(M(C,D))$ is obtained*
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