ON THE EXTENDABILITY OF QUAIDIVISIBLE OPTIMAL ARCS

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1. Divisible and Quasidivisible Arcs

◇ A multiset in $\mathrm{PG}(k-1,q)$ is a mapping

$$\mathcal{K}: \begin{cases} \mathcal{P} & \rightarrow & \mathbb{N}_0, \\ P & \rightarrow & \mathcal{K}(P). \end{cases}$$

◇ $\mathcal{K}(P)$ – multiplicity of the point P.

◇ $\mathcal{Q} \subset \mathcal{P}$: $\mathcal{K}(\mathcal{Q}) = \sum_{P \in \mathcal{Q}} \mathcal{K}(P)$ – multiplicity of the set $\mathcal{Q}$.

◇ $\mathcal{K}(\mathcal{P})$ – the cardinality of $\mathcal{K}$.

◇ Points, lines, ... ,hyperplanes of multiplicity i are called i-points, i-lines, ... , i-hyperplanes.

◇ a_i – the number of hyperplanes H with $\mathcal{K}(H) = i$

◇ $(a_i)_{i \geq 0}$ – the spectrum of $\mathcal{K}$

Definition. (n, w)-arc in $\mathrm{PG}(k-1, q)$: a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane H: $\mathcal{K}(H) \leq w$;

3) there exists a hyperplane H_0: $\mathcal{K}(H_0) = w$.

Definition. (n, w)-blocking set in $\mathrm{PG}(k-1, q)$

(or (n, w)-minihyper): a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane H: $\mathcal{K}(H) \geq w$;

3) there exists a hyperplane H_0: $\mathcal{K}(H_0) = w$.

Definition. An (n, w)-arc $\mathcal{K}$ in $\mathrm{PG}(k-1, q)$ is called t-extendable, if there exists an $(n+t, w)$-arc $\mathcal{K}'$ in $\mathrm{PG}(k-1, q)$ with $\mathcal{K}'(P) \geq \mathcal{K}(P)$ for every point $P \in \mathcal{P}$. An 1-extendable arc is called extendable.

Definition. An arc $\mathcal{K}$ in $\mathrm{PG}(k-1, q)$ with $\mathcal{K}(\mathcal{P}) = n$ and spectrum (a_i) is said to be divisible with divisor Δ, $\Delta > 1$, if $a_i = 0$ for all $i \not\equiv n \pmod{\Delta}$.

Definition. An arc $\mathcal{K}$ with $\mathcal{K}(\mathcal{P}) = n$ and spectrum (a_i) is said to be t-quasidivisible with divisor Δ, $\Delta > 1$, (or t-quasidivisible modulo Δ) if $a_i = 0$ for all $i \not\equiv n, n+1, \ldots, n+t \pmod{\Delta}$.

2. Linear codes as multisets of points

$[n,k,d]_q$-code C of full length	$\Leftrightarrow$	$(n, w = n-d)$-arc $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$
$\mathbf{0} \neq \boldsymbol{u} \in C$, $\mathrm{wt}(\boldsymbol{u}) = u$	$\Leftrightarrow$	a hyperplane H with $\mathcal{K}(H) = n - u$,
extendable $[n,k,d]_q$-code C	$\Leftrightarrow$	extendable $(n, n-d)$-arc $\mathcal{K}$
divisible $[n,k,d]_q$-code $A_i = 0$ for all $i \not\equiv 0 \pmod{\Delta}$	$\Leftrightarrow$	divisible $(n, n-d)$-arc in $\mathrm{PG}(k-1,q)$ $a_i = 0$ for all $i \not\equiv n \pmod{\Delta}$

◇ Griesmer bound: Let $\mathcal{C}$ be an $[n,k,d]_q$-code. Then

$$n_q(k,d) \geq g_q(k,d) = \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil$$

◇ Griesmer arcs: arcs associated with codes meeting the Griesmer bound

Griesmer $[n,k,d]_q$ codes $\Leftrightarrow$ Griesmer (n,w)-arcs in $\mathrm{PG}(k-1,q)$

$n = \sum_{i=0}^{k-1} \lceil d/q^i \rceil$ $\qquad$ $n = \sum_{i=0}^{k-1} \lceil (n-w)/q^i \rceil$

Theorem. (R. Hill, P. Lizak, 1995, geometric version) Let $\mathcal{K}$ be a (n, w)-arc in $\mathrm{PG}(k-1, q)$ with $\gcd(n-w, q) = 1$. Let further $\mathcal{K}(H) \equiv n$ or $w \pmod q$ for all hyperplanes H. Then $\mathcal{K}$ is extendable to a divisible $(n+1, w)$-arc in $\mathrm{PG}(k-1, q)$. In particular, every 1-quasidivisible arc with divisor q is extendable.

Theorem. (T. Maruta, 2004, geometric version) Let $\mathcal{K}$ be a 2-quasidivisible (n, w)-arc in $\mathrm{PG}(k-1, q)$, $q \geq 5$, odd, with divisor q. Then $\mathcal{K}$ is extendable to an $(n+1, w)$-arc in $\mathrm{PG}(k-1, q)$.

3. A New Extension Results

$\diamond$ $\mathcal{K}$ - (n, w)-arc in $\Sigma = \mathrm{PG}(k-1, q)$

$\diamond$ for every hyperplane H, we have $\mathcal{K}(H) \equiv n, n+1, \ldots, n+t \pmod{q}$ where $0 < t < q$ is an integer constant, i.e. $\mathcal{K}$ is t-quasidivisible modulo q.

$\diamond$ Define an arc $\widetilde{\mathcal{K}}$ in the dual space $\widetilde{\Sigma}$

$$(\star) \qquad \widetilde{\mathcal{K}} : \begin{cases} \mathcal{H} & \rightarrow & \mathbb{N}_0, \\ H & \rightarrow & \widetilde{\mathcal{K}}(H) := n + t - \mathcal{K}(H) \pmod{q}. \end{cases}$$

where $\mathcal{H}$ is the set of all hyperplanes of Σ.

Theorem. Let $\mathcal{K}$ be an (n, w)-arc in $\Sigma = \mathrm{PG}(k-1, q)$ which is t-quasidivisible modulo q, $t < q$. Let

$$\widetilde{\mathcal{K}} = \sum_{i=1}^{c} \chi_{\widetilde{H}_i} + \widetilde{\mathcal{K}}'$$

for some arc $\widetilde{\mathcal{K}'}$ and c not necessarily different hyperplanes $\widetilde{H}_1, \ldots, \widetilde{H}_c$ then $\mathcal{K}$ is c-extendable. In particular, if $\widetilde{\mathcal{K}}$ contains a hyperplane in its support then $\mathcal{K}$ is extendable.

Note: the above theorem is a sufficient but not a necessary condition.

Theorem. Let $\widetilde{S}$ be a subspace of $\widetilde{\Sigma}$ then $\widetilde{\mathcal{K}}(\widetilde{S}) \equiv t \pmod q$.

$\diamond$ The arc $\widetilde{\mathcal{K}}$ has the following properties:

- the multiplicity of each point is at most t;

- each subspace $\widetilde{S}$ of dimension r, $1 \leq r \leq k-1$, is of multiplicity

$$\widetilde{\mathcal{K}}(\widetilde{S}) \geq t v_r,$$

where $v_r = \dfrac{q^r - 1}{q - 1}$.

◇ Consider a Griesmer (n, w)-arc $\mathcal{K}$ $(w = n - d)$ in $\mathrm{PG}(k-1, q)$ with

$$d = sq^{k-1} - \varepsilon_{k-2}q^{k-2} - \ldots - \varepsilon_1 q - \varepsilon_0,$$

◇ Set $w_i :=$ maximal multiplicity of a subspace of codimension i. Then:

$$\begin{array}{rcl}
n & = & sv_k - \varepsilon_{k-2}v_{k-1} - \ldots - \varepsilon_2 v_3 - \varepsilon_1 v_2 - \varepsilon_0 v_1, \\
w_1 & = & sv_{k-1} - \varepsilon_{k-2}v_{k-2} - \ldots - \varepsilon_2 v_2 - \varepsilon_1 v_1, \\
w_2 & = & sv_{k-2} - \varepsilon_{k-2}v_{k-3} - \ldots - \varepsilon_2 v_1, \\
\vdots & \vdots & \vdots \\
w_{k-2} & = & sv_2 - \varepsilon_{k-2}v_1, \\
w_{k-1} & = & sv_1.
\end{array}$$

Lemma. Let $\mathcal{K}$ be a Griesmer $(n, w = n - d)$-arc with d as above, which is t-quasidivisible modulo q, i.e. $\mathcal{K}(H) \equiv n, n+1, \ldots, n+t \pmod{q}$ for every hyperplane H. Let S be a hyperline (subspace of codimension 2) in a hyperplane H_0 with $\mathcal{K}(H_0) = w_1 - aq$ where $a \geq 0$ is an integer.

(i) If $\mathcal{K}(S) = w_2 - a - b$, $0 \leq b \leq t-2$, then $\widetilde{\mathcal{K}}(\widetilde{S}) \leq t + bq$;

(ii) If $\mathcal{K}(S) = w_2 - a - b$, $b \geq t-1$, then $\widetilde{\mathcal{K}}(\widetilde{S}) \leq t + (t-1)q$.

Lemma. Let $\mathcal{K}$ be a t-quasidivisible Griesmer (n, w)-arc in $\mathrm{PG}(k-1, q)$, and let $\widetilde{\mathcal{K}}$ be as in $(\star)$. Let T be a subspace of codimension 3 in $\mathrm{PG}(k-1, q)$ with $\mathcal{K}(T) = w_3$. Then $\widetilde{\mathcal{K}}(\widetilde{T}) \leq t(q+1) + \varepsilon_1 q$.

Lemma. Let $\mathcal{K}$ be a t-quasidivisible Griesmer (n, w)-arc in $\mathrm{PG}(k-1, q)$, $q \geq 3$ with

$$d = n - w = sq^{k-1} - \sum_{i=0}^{k-2} \varepsilon_i q^i, \quad (\varepsilon_0 = t)$$

and let $\widetilde{\mathcal{K}}$ be defined as in $(\star)$. Let further $\varepsilon_0, \varepsilon_1 < \sqrt{q}$. For every maximal subspace T of codimension 3 in $\mathrm{PG}(k-1, q)$, i.e. a subspace with $\mathcal{K}(T) = w_3$, it holds

$$\widetilde{\mathcal{K}}(\widetilde{T}) = t(q+1).$$

Theorem. Let $\mathcal{K}$ be t-quasidivisible Griesmer (n, w)-arc in $\mathrm{PG}(k-1, q)$, $q \geq 3$ with

$$d = n - w = sq^{k-1} - \sum_{i=0}^{k-2} \varepsilon_i q^i.$$

Let $\widetilde{\mathcal{K}}$ be defined as in $(\star)$. Let U be a subspace in $\mathrm{PG}(k-1, q)$ of maximal multiplicity w_r with $\operatorname{codim} U = r$, $1 \leq r \leq k$ (if $\operatorname{codim} U = k$, $U = \varnothing$). If $t = \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{r-2} < \sqrt{q}$, then

$$\widetilde{\mathcal{K}}(\widetilde{U}) = tv_{r-1}.$$

In particular, $\widetilde{\mathcal{K}}(\widetilde{\Sigma}) = tv_{k-1}$.

Theorem. (I. Landjev, P. Vanderdriesche, 2012) If $t \leq q - q/p$ any (tv_{k-1}, tv_{k-2})-minihyper in $\mathrm{PG}(k-1, q)$ is a sum of t hyperplanes.

Theorem. (main theorem)

Let $\mathcal{K}$ be a t-quasidivisible Griesmer arc in $\mathrm{PG}(k-1, q)$ with parameters $(n, n-d)$, where

$$d = sq^{k-1} - \sum_{i=0}^{k-2} \varepsilon_i q^i.$$

Let $\varepsilon_0 = t, \ldots \varepsilon_{k-2} < \sqrt{q}$. Then $\mathcal{K}$ is t-extendable.

Proof.

◇ By the previous theorem $\widetilde{\mathcal{K}}$ has parameters (tv_{k-1}, tv_{k-2}).

◇ By a theorem by I. Landjev and P. Vandendriesche, $\widetilde{\mathcal{K}}$ is the sum of t (not necessarily different) hyperplanes.

◇ Hence $\widetilde{\mathcal{K}}$ is t-extendable.

4. What is this good for?

d	$g_5(4,d)$	$n_5(4,d)$	$\mathcal{K}$	$\mathcal{K}\|_H$
81	103	103–104	$(103, 22)$	$(22, 5)$-arc
82	104	104–105	$(104, 22)$	in $\mathrm{PG}(2, 5)$
161	203	203–204	$(203, 42)$	$(42, 9)$-arc
162	204	204–205	$(204, 42)$	in $\mathrm{PG}(2, 5)$

$$d = 82 = 5^3 - 5^2 - 3 \cdot 5 - 3$$

$$s = 1, \varepsilon_2 = 1, \varepsilon_1 = 3, \varepsilon_0 = t = 3;$$

$$\varepsilon_0, \varepsilon_1 \geq \sqrt{q}$$

$$w_3 = 1, w_2 = 5, w_1 = 22, w_0 = n = 104$$

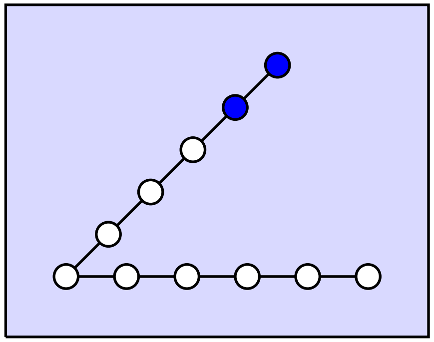
Let $\mathcal{K}$ be a $(104, 22)$-arc in $\mathrm{PG}(3, 5)$

Some facts about $\mathcal{K}$ and $\widetilde{\mathcal{K}}$:

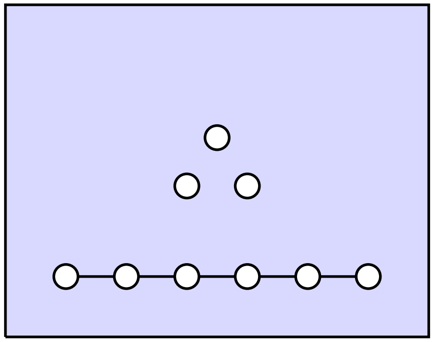
- $\mathcal{K}$ is Griesmer 3-quasidivisible projective arc;
- all planes have multiplicity ≥ 14;
- the structure of $\mathcal{K}|_H$, where H is a maximal plane, is known: $(22, 5)$-arcs;

$(22,5)$-arcs in $\mathrm{PG}(2,5)$

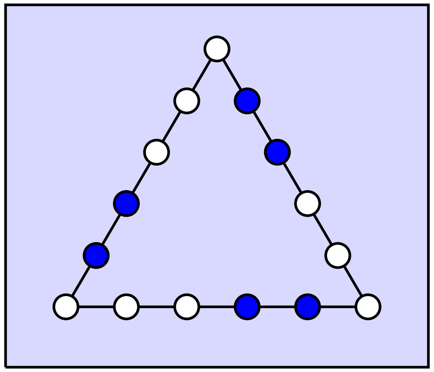
(A)



(B)



(C)



(A) $a_5 = 14, a_4 = 15, a_2 = 1, a_0 = 1$

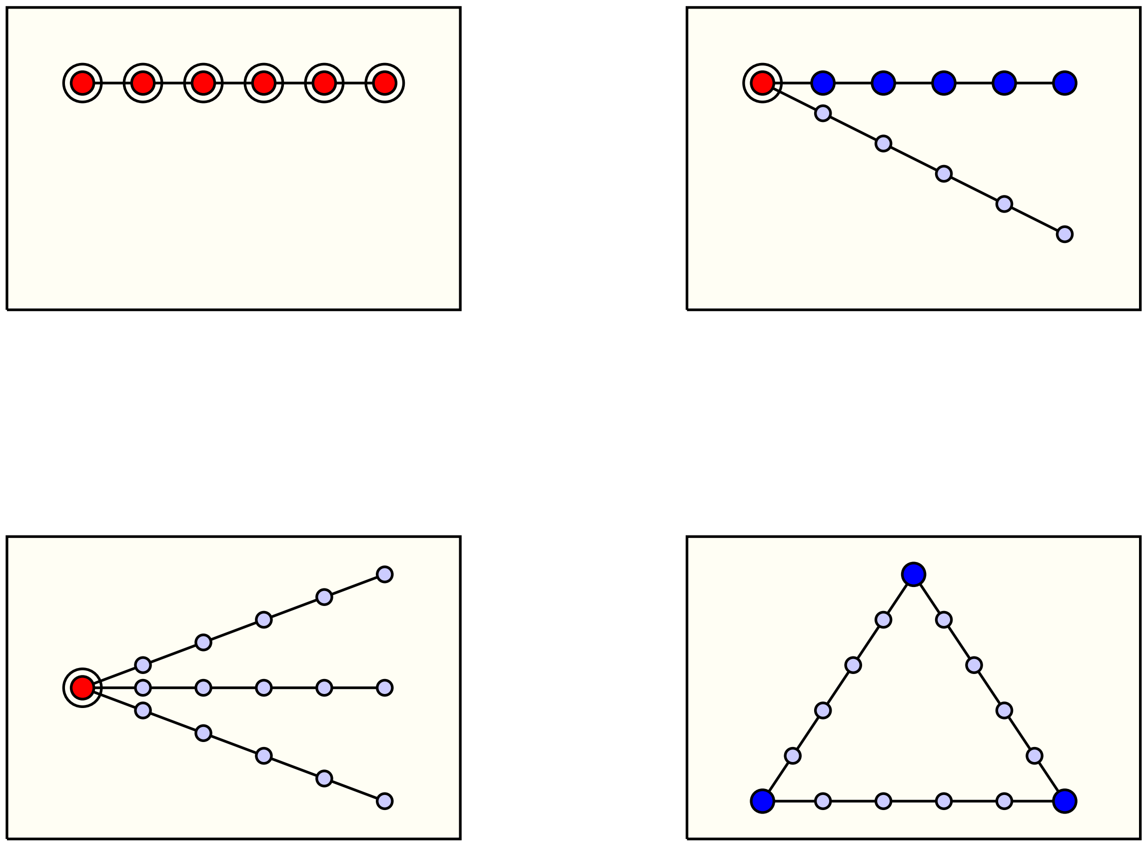
(B) $a_5 = 15, a_4 = 12, a_3 = 3, a_0 = 1$

(C) $a_5 = 18, a_4 = 6, a_3 = 4, a_2 = 3$

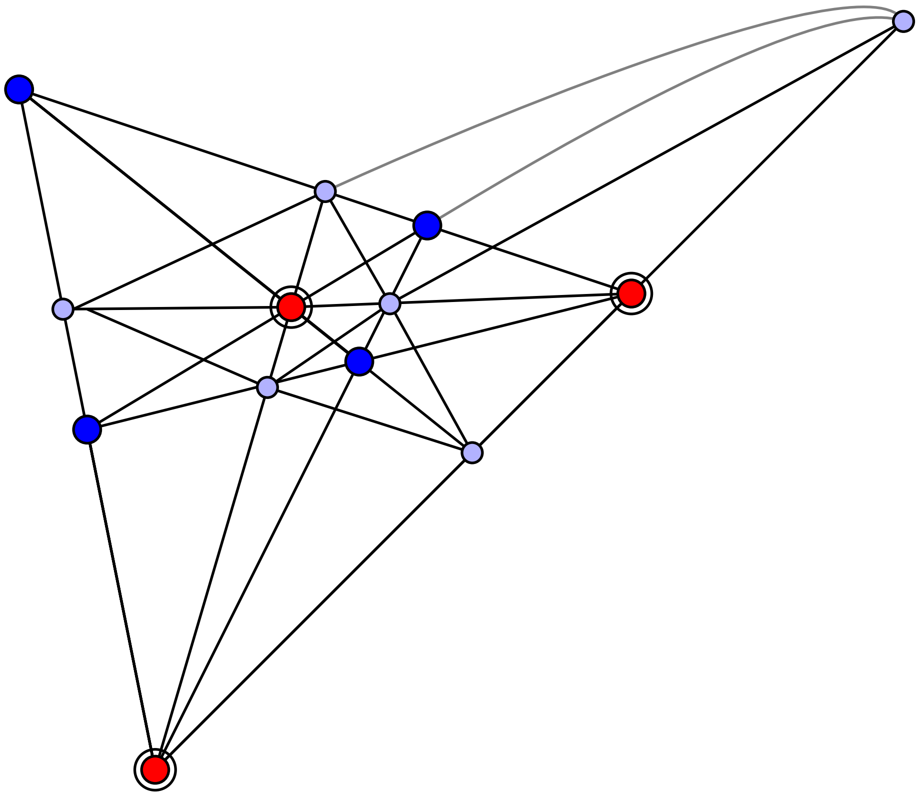
N.B. $(22, 5)$-arcs do not have 1-lines!

The arcs $\widetilde{\mathcal{K}}|_{\widetilde{H}}$

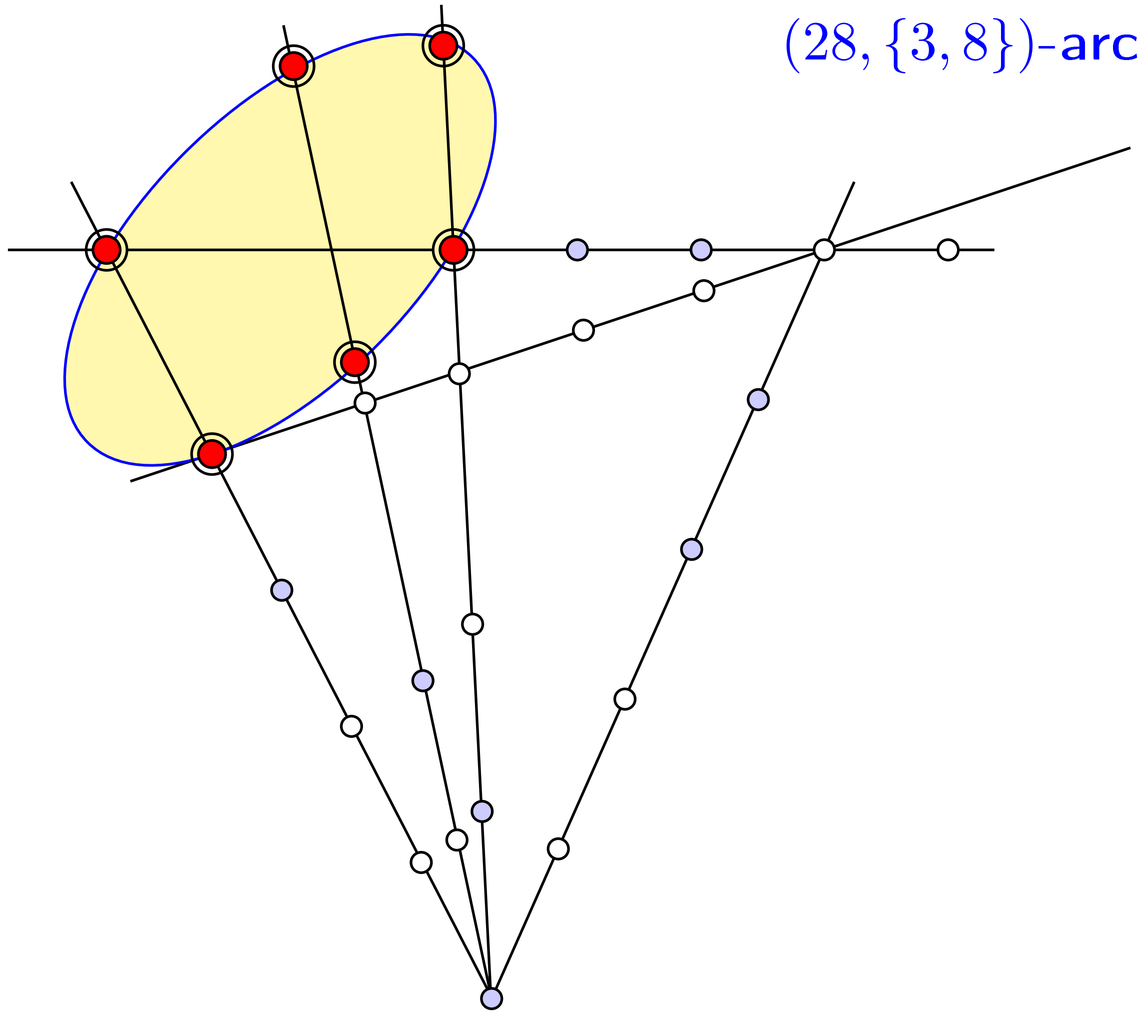
$(18, \{3, 8, 13, 18\})$-arcs



$(23, \{3, 8\})$-arc



$(28, \{3, 8\})$-**arc**



- From the structure of the maximal planes with respect to $\mathcal{K}$:

$$|\widetilde{\mathcal{K}}| = 93,\ 118 \text{ or } 143.$$

- $|\widetilde{\mathcal{K}}| = 118$ and 143 are ruled out investigating the structure of $\mathcal{K}$

- Hence $|\widetilde{\mathcal{K}}| = 93$ and $\widetilde{\mathcal{K}}$ is a sum of three planes.

- Hence $\mathcal{K}$ is 3-extendable to a (non-existent) $(107, 22)$-arc.

- There is no $[104, 4, 82]_5$-code and

$$n_5(4, 82) = 105.$$