Covering of F_3^n with spheres of maximal radius

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The football pool problem

Find the minimum cardinality of a mixed binary/ternary code of covering radius $R = 1, 2, 3$ or 4.

Bounds on $K(4; b; R)$

t	b	$R = 1$	$R = 2$	$R = 3$	$R = 4$
4	**1**	**18**	**6**	**3**	**2**
4	**2**	**36**	**10**	**4**	**3**
4	**3**	**60-72**	**15-18**	**6**	**3**
4	**4**	**107-128**	**20-24**	**10**	**4**
4	**5**	**195-238**	**32-48**	**12-16**	**6**
4	**6**	**356-432**	**55-72**	**16-24**	**8-10**
4	**7**	**672-852**	**93-144**	**22-40**	**10-15**
4	**8**	**1257-1296**	**168-252**	**36-60**	**12-22**
4	**9**	**2370-2592**	**290-480**	**58-107**	**16-36**
4	**10**	**4366-5184**	**494-852**	**91-183**	**24-60**

Bounds on $K(5; b; R)$

t	b	$R = 1$	$R = 2$	$R = 3$	$R = 4$
5	**1**	**45-54**	**12**	**4**	**3**
5	**2**	**80-96**	**16-21**	**7**	**3**
5	**3**	**148-168**	**24-36**	**11-12**	**4**
5	**4**	**268-324**	**42-64**	**13-21**	**7**
5	**5**	**509-624**	**71-108**	**18-32**	**8-12**
5	**6**	**936-1184**	**126-192**	**28-54**	**11-18**
5	**7**	**1791-1944**	**222-348**	**44-86**	**12-29**
5	**8**	**3353-3888**	**385-664**	**76-144**	**19-48**
5	**9**	**6221-7776**	**669-1224**	**117-245**	**30-79**

INVERSE FOOTBALL POOL PROBLEM

Cover the space F_3^n with minimum number of spheres of maximal radius.

$T(n)$ – the minimum cardinality of a ternary code of length n such that the spheres centered at the codewords of radius n cover F_3^n.

The sequence $T(n)$ is a part of The on-line encyclopedia of integer sequences, number A086676.

Main Problems

- **Find the value of $T(n)$;**
- **Find all optimal coverings.**

What is in this talk?

- **General approach of finding exact value or bounds on $T(n+1)$ when $T(n)$ and all optimal coverings of F_3^n are known;**

- **Combinatorial proof of known computer based results for $n \leq 7$.**

Known results for $T(n)$ for $1 \leq n \leq 13$

n	$T(n)$
1	**2**
2	**3**
3	**5**
4	**8**
5	**12**
6	**18**
7	**29**
8	**44**
9	**68**
10	**102–104**
11	**153–172**
12	**230–264**
13	**345–408**

Let C be a covering of F_3^{n+1}.

Denote by C_i^k the set of all codewords from C having i in k-th coordinate without this coordinate.

It is clear that for $i, j \in \{0, 1, 2\}$, $i \neq j$ and for any $k, 1 \leq k \leq n+1$ the set $C_i^k \cup C_j^k$ is a covering of F_3^n.

A straightforward recursive bound on $T(n)$ is given by

$$T(n+1) \geq \left\lceil \frac{3}{2} T(n) \right\rceil .$$

Proposition.

If $T(n+1) = \frac{3}{2}T(n)$ **and the minimum distance of all optimal coverings of** F_3^n **equals** t **then the minimum distance of all optimal covering of** F_3^{n+1} **equals** $t+1$**.**

Denote by a_k the number of unordered pairs (u, v), $u, v \in C$ such that $d(u, v) = k$.

The set $\{a_1, a_2, \ldots, a_n\}$ is referred to as pair distance distribution of C.

For each k, $1 \leq k \leq n$ consider a graph G_k with vertices the codewords of C. Two vertices u and v are connected with an edge if and only if $d(u, v) = k$.
Call this graph induced graph of C of weight k.

Proposition.

Suppose $T(n)$ is even and there exists a unique optimal covering of F_3^n with pair distance distribution $\{a_1, a_2, \ldots, a_n\}$. If there exists k such that $a_k \neq 0$, $a_{k-1} = 0$ and the induced graph G_k has an odd cycle then $T(n+1) > \frac{3}{2}T(n)$.

Let $T(n) = 2t$ **and assume** $T(n+1) = \frac{3}{2}T(n) = 3t$**. It follows that** $c_0 = c_1 = c_2 = t$ **and the set** $C_0 \cup C_1$ **is the optimal covering of** F_3^n**.**

We prove that if $u, v \in C_0 \cup C_1$ **are such that** $d(u, v) = k$ **then** $u \in C_0, v \in C_1$ **or** $u \in C_1, v \in C_0$**.**

Indeed, assume $u, v \in C_i$ **for** $i = 0$ **or** 1 **and let**

$$u = (u_2, \ldots, u_{n+1}), \quad v = (v_2, \ldots, v_{n+1}).$$

Without loss of generality assume $u_2 = 0$ **and** $v_2 = 1$**. Since** $C_0^2 \cup C_1^2$ **is equivalent to the unique optimal covering of** F_3^n**,** $u' = (i, u_3, \ldots, u_{n+1}), v' = (i, v_3, \ldots, v_{n+1}) \in C_0^2 \cup C_1^2$ **and** $d(u', v') = d(u, v) - 1 = k - 1$ **we get a contradiction with** $a_{k-1} = 0$**.**

Hence, if two vertices u and v of G_k are connected with an edge then $u \in C_0, v \in C_1$ or $u \in C_1, v \in C_0$. This is impossible for the elements of an odd cycle in G_k, a contradiction. Therefore $T(n+1) > 3t = \frac{3}{2}T(n)$.

Proposition

It is true that:

$T(2) = 3$; $\quad T(3) = 5$; $\quad T(4) = 8$;

$T(5) = 12$; $\quad T(6) = 18$.

and for every n, $2 \leq n \leq 6$ **there exists unique optimal covering of** F_3^n.

The first two cases $T(2) = 3$ and $T(3) = 5$ are straightforward. The corresponding unique optimal coverings are given by

$$\mathcal{C}_2 = \{00, 11, 22\} \textbf{ and } \mathcal{C}_3 = \{000, 110, 101, 011, 222\}.$$

It follows from $T(3) = 5$ that $T(4) \geq 8$. Let $\mathcal{C}_4$ be a covering of F_3^4 with cardinality 8. Since $T(3) = 5$ we may assume that $c_0 = c_1 = 3$ and $c_2 = 2$. Therefore both $C_0 \cup C_2$ and $C_1 \cup C_2$ are equivalent to $\mathcal{C}_3$.

Observing the structure of $\mathcal{C}_3$ we conclude that up to equivalence there are two choices for $C_2 - \{000, 222\}$ or $\{000, 011\}$. The corresponding options for C_1 are: $\{110, 101, 011\}$ and $\{110, 101, 222\}$. In the first case there are two possible choices for C_0:

$$\{110, 101, 011\}, \{112, 121, 211\},$$

both do not result in a covering.

In the second case there are also two possible choices for C_0: $\{110, 101, 222\}$ or $\{122, 210, 201\}$. The second one gives a covering. Therefore, up to equivalence there exists a unique covering of F_3^4:

$$\{0122, 0210, 0201, 1222, 1110, 1101, 2000, 2011\}.$$

The above covering is equivalent to:

$$\mathcal{C}_4 = \{0122, 0000, 0011, 1022, 1100, 1111, 2210, 2201\}.$$

The pair distance distribution of $\mathcal{C}_4$ **is given by** $a_1 = 0$**,** $a_2 = 6$**,** $a_3 = 16$ **and** $a_4 = 6$**.**

The unique optimal covering of F_3^6:

1.	0 0 0 0 0 0
2.	2 1 2 1 0 0
3.	1 2 2 0 1 0
4.	2 0 1 2 1 0
5.	0 2 1 1 2 0
6.	1 1 0 2 2 0
7.	2 2 1 0 0 1
8.	1 0 2 2 0 1
9.	1 1 1 1 1 1

10.	0 2 0 2 1 1
11.	0 1 2 0 2 1
12.	2 0 0 1 2 1
13.	1 2 0 1 0 2
14.	0 1 1 2 0 2
15.	2 1 0 0 1 2
16.	0 0 2 1 1 2
17.	1 0 1 0 2 2
18.	2 2 2 2 2 2

The pair distance distribution of $\mathcal{C}_6$ is given by

$$a_4 = 135,\ a_6 = 18.$$

Proposition

It is true that $T(7) = 29$.

Suppose $T(7) \leq 28$ **and consider a covering** C **of** F_3^7 **with 28 elements. Since** $T(6) = 18$ **we have that for any** $t = 1, 2, \ldots, 7$ **and for any two** $i, j \in \{0, 1, 2\}$ **it is true that** $c_i^t + c_j^t \geq 18$**. It follows from** $c_0^t + c_1^t + c_2^t = 28$ **that for any** $t = 1, 2, \ldots, 7$ **there exist** $i, j \in \{0, 1, 2\}$ **such that** $c_i^t + c_j^t = 18$**. Hence,** $C_i^t \cup C_j^t \equiv \mathcal{C}_6$**.**

Without loss of generality when $t = 1$ assume $i = 0, j = 1$. Consider three codewords

$$\begin{array}{rcl} u &=& (i, \ u_2, \ u_3, \ \ldots, \ u_7) \\ v &=& (i, \ v_2, \ v_3, \ \ldots, \ v_7) \\ u &=& (i, \ w_2, \ w_3, \ \ldots, \ w_7) \end{array}$$

for $i = 0$ or 1. Since $C_0 \cup C_1 \equiv \mathcal{C}_6$ we have that all pairwise distances between u, v, w equal 4 or 6. Assume that for some t we have $\{u_t, v_t, w_t\} = \{0, 1, 2\}$. Without loss of generality $t = 2$. All pairwise distances between the vectors $(i, u_3, \ldots, u_7)$, $(i, v_3, \ldots, v_7)$, $(i, w_3, \ldots, w_7)$ equal 3 or 5, a contradiction to the fact that two of them are elements of $\mathcal{C}_6$.

Without loss of generality let $000000, 111111 \in C_0$**. Since all elements of** C_6 **contain at least one 2 it follows from the above observations that** $C_1 = \mathcal{C}_6 \backslash C_0$**. It is obvious that there exist a 0,1,2 coordinate in** C_1**, a contradiction.**

Therefore $T(7) \geq 29$ **and since there exists a covering of** F_3^7 **of cardinality 29, we conclude that** $T(7) = 29$**.**

Suppose we know $T(n)$ and all optimal coverings of F_3^n. Using the above approach we are able:

- **for any $T(n)$ (even or odd) to determine whether $T(n+1) = \left\lceil \frac{3}{2} T(n) \right\rceil$ and if so, to find all optimal coverings;**
- **for $T(n)$ even to determine whether $T(n+1) = \left\lceil \frac{3}{2} T(n) \right\rceil + 1$ and if so, to find all optimal coverings.**

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