On a Construction of Optimal Codes in Term Rank Metric via $p(x)$-circulants

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Outline

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Matrix channel

Fix

- $\mathbb{F}_q$ – finite field, $q = p^a$, $a \in \mathbb{N}$
- m, $n \in \mathbb{N}$, $m \leq n$.

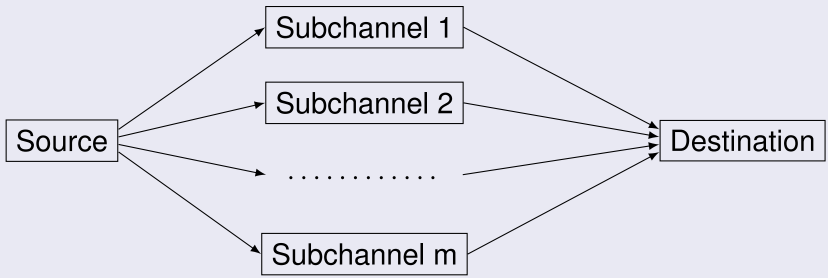
Matrix channel $\mathcal{W}$

Matrix channel

Fix

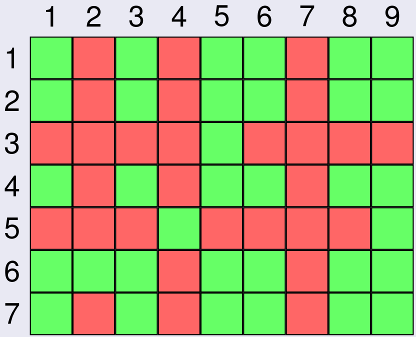
- $\mathbb{F}_q$ – finite field, $q = p^a$, $a \in \mathbb{N}$
- m, $n \in \mathbb{N}$, $m \leq n$.

Matrix channel $\mathcal{W}$



Crisscross errors

Crisscross error pattern ($m = 7$, $n = 9$)



Matrix channel applications

- Data storage systems (e.g. memory chips)
- Magnetic tapes
- Wireless communications
- Space-time coding
- Random network coding
- Public-key cryptography
- Steganography

Outline

1. Preliminaries
   - Matrix channel with crisscross errors
   - Term rank metric and rank metric spaces
   - Linear codes in $\mathcal{M}_{TR}$ and $\mathcal{M}_R$
2. Algebra of $p(x)$-circulants and a code construction
   - $p(x)$-circulants
   - Code construction
3. Some optimal codes
   - Optimal codes in $\mathcal{M}_R$
   - Optimal codes in $\mathcal{M}_{TR}$
   - Open questions

Term rank metric

Definition

Let $\mathbb{F}_q^{m\times n}$ be a set of all matrices over $\mathbb{F}_q$ with m rows and n columns.

Term rank weight

$$\forall A \in \mathbb{F}_q^{m\times n} : \ \|A\|_{TR} = \min_{\mathcal{I}(A)} |\mathcal{I}(\mathcal{A})|,$$

$\mathcal{I}(A)$ is a set of lines (rows and/or columns) in A which cover all nonzero elements of A

Term rank metric

Example

$$\left\| \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\|_{TR} = 1, \quad \left\| \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\|_{TR} = 2$$

Example

$$\left\| \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\|_{TR} = 4, \quad \left\| \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right\|_{TR} = 4$$

Term rank metric

Example

$$\left\| \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\|_{TR} = 1, \quad \left\| \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\|_{TR} = 2$$

Example

$$\left\| \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\|_{TR} = 4, \quad \left\| \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right\|_{TR} = 4$$

Term rank metric

Term rank distance function

$$d_{TR}(A, B) = \|A - B\|_{TR}$$

Term rank metric space

$$\mathcal{M}_{TR} = (\mathbb{F}_q^{m \times n}, d_{TR})$$

Term rank metric

Definition

A diagonal Δ_τ in a matrix $A \in \mathbb{F}_q^{m\times n}$ is a set of positions

$$\Delta_\tau = \{(0, \tau(0)), (1, \tau(1)), \ldots, (m-1, \tau(m-1))\},$$

where τ is an injection from $[0,\ m-1]$ to $[0,\ n-1]$.

Theorem

$$\forall A \in \mathbb{F}_q^{m\times n}:\ \|A\|_{TR} = \max_\tau |\Delta_\tau(A)|,$$

where τ runs overall injections from $[0,\ m-1]$ to $[0,\ n-1]$ and $|\Delta_\tau(A)|$ – a number of nonzero entries of A on the Δ_τ.

Rank metric

Rank weight

$$\forall A \in \mathbb{F}_q^{m\times n} : \ \|A\|_R = \mathrm{rank}(A)$$

Rank distance function

$$d_R(A, B) = \mathrm{rank}(A - B)$$

Rank metric space

$$\mathcal{M}_R = (\mathbb{F}_q^{m\times n}, d_R)$$

Metric spaces

Connection between term rank metric and rank metric

$$\forall A,\ B \in \mathbb{F}_q^{m\times n}:\ d_R(A,B) \le d_{TR}(A,B) \le \min\{m,n\}$$

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   - Optimal codes in $\mathcal{M}_{TR}$
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Linear codes

Definition

A linear $[m \cdot n, k]_q$-code $\mathcal{C}$ is a k-dimensional subspace of $\mathbb{F}_q^{m \times n}$.

Minimal distances of $\mathcal{C}$

$$D_{TR}(\mathcal{C}) \triangleq \min_{A \in \mathcal{C} \setminus \{0\}} \|A\|_{TR}, \quad D_R(\mathcal{C}) \triangleq \min_{A \in \mathcal{C} \setminus \{0\}} \|A\|_R.$$

Connection between minimal distances

$$D_R(\mathcal{C}) \leq D_{TR}(\mathcal{C}) \leq \min\{m, n\}$$

Optimal linear codes

Singleton-type bound

For any $[m \cdot n, k]_q$-code $\mathcal{C}$ we have

$$k \leq n(m - D_{TR}(\mathcal{C}) + 1), \quad k \leq n(m - D_{TR}(\mathcal{C}) + 1).$$

Definition

A linear code $\mathcal{C}$ is called *optimal* in $\mathcal{M}_{TR}$ (in $\mathcal{M}_R$) if $k = n(m - D_{TR}(\mathcal{C}) + 1)$ (resp. $k = n(m - D_R(\mathcal{C}) + 1)$).

Constructions of optimal codes in $\mathcal{M}_{TR}$

- By construct an optimal codes in $\mathcal{M}_R$ (e.g. Gabidulin codes)
- By considering a set of all circulant $m \times n$ matrices over $\mathbb{F}_q$ (only for the case $D_{TR} = m$)
- By considering a set of Toeplic-like matrices (only for the case $D_{TR} = 2$)
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$p(x)$-circulants

Fix

A monic polynomial $p(x) \in \mathbb{F}_q[x]$ of degree n

$$p(x) = x^n - p_{n-1}x^{n-1} - \ldots - p_1 x - p_0,$$

$p_0 \neq 0$, with its companion matrix

$$B_{p(x)} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ p_0 & p_1 & p_2 & \ldots & p_{n-1} \end{pmatrix} \in \mathbb{F}_q^{n\times n}.$$

$p(x)$-circulants

Define $\mathbb{F}_q$-algebras homomorphism

$$\varphi : \mathbb{F}_q[x] \to \mathbb{F}_q^{n\times n}, \quad x^i \mapsto (B_{p(x)})^i$$

Properties of φ

- $\ker \varphi = (p(x))$
- $\mathcal{C}_{p(x)} \triangleq \mathrm{Im}\,\varphi \simeq \mathbb{F}_q[x]/(p(x))$
- $\dim \mathcal{C}_{p(x)} = n$

Definition

Elements of $\mathcal{C}_{p(x)}$ we called $p(x)$-circulants

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$p(x)$-circulants

Example ($p(x) = x^n - 1$)

C_{x^n-1} is the algebra of all circulant $(n \times n)$-matrices over $\mathbb{F}_q$

Example ($p(x) = x^5 + 1$)

C_{x^5+1} is the algebra of all skew-circulant (negacirculant)

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ -a_4 & a_0 & a_1 & a_2 & a_3 \\ -a_3 & -a_4 & a_0 & a_1 & a_2 \\ -a_2 & -a_3 & -a_4 & a_0 & a_1 \\ -a_1 & -a_2 & -a_3 & -a_4 & a_0 \end{pmatrix}$$

$p(x)$-circulants

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$p(x)$-circulants

Example ($p(x) = x^5 - r$)

C_{x^5-r} is the algebra of all r-circulant matrices

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ ra_4 & a_0 & a_1 & a_2 & a_3 \\ ra_3 & ra_4 & a_0 & a_1 & a_2 \\ ra_2 & ra_3 & ra_4 & a_0 & a_1 \\ ra_1 & ra_2 & ra_3 & ra_4 & a_0 \end{pmatrix}$$

$p(x)$-circulants

Example ($p(x) = x^5 - x - 1$)

C_{x^5-x-1} is the algebra of all *FLS*-circulant matrices

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_0 + a_4 & a_1 & a_2 & a_3 \\ a_3 & a_4 + a_3 & a_0 + a_4 & a_1 & a_2 \\ a_2 & a_3 + a_2 & a_4 + a_3 & a_0 + a_4 & a_1 \\ a_1 & a_2 + a_1 & a_3 + a_2 & a_4 + a_3 & a_0 + a_4 \end{pmatrix}$$

Facts on $p(x)$-circulants

$p(x)$-circulant is defined by its first row

$$\mu_{p(x)} : \mathbb{F}_q^n \to \mathcal{C}_{p(x)}, \quad (a_0, \dots, a_{n-1}) \mapsto \varphi\left(a_0 + \dots + a_{n-1}x^{n-1}\right)$$

Some useful facts

- $A = \mu_{p(x)}((a_0, \dots, a_{n-1})) \in \mathcal{C}_{p(x)}$ is invertible iff

$$\gcd(a_0 + a_1 x + \dots + a_{n-1}x^{n-1}, p(x)) = 1$$

- $\mathcal{C}_{p(x)}$ is a field iff $p(x)$ is irreducible

Outline

Codes via $p(x)$-circulants

Definition

$\mathcal{C}_{p(x)}$ is a linear $[n \cdot n,\ n]_q$-code in $\mathcal{M}_{TR}$ and $\mathcal{M}_R$

Code shortening

To construct a linear $[m \cdot n,\ n]_q$-code for each $m \in [1, n]$ and any $\mathcal{J} \subseteq [0, n-1]$, $|\mathcal{J}| = n - m$, define the $\mathbb{F}_q$-linear "shortening" map

$$\sigma_{\mathcal{J}}^{(n-m)} : \mathbb{F}_q^{n \times n} \to \mathbb{F}_q^{m \times n}$$

by cut all rows with indexes from $\mathcal{J}$.

Codes via $p(x)$-circulants

Definition

$\sigma_{\mathcal{J}}^{(n-m)}(\mathcal{C}_{p(x)})$ is a linear $[m \cdot n,\ n]_q$-code in $\mathcal{M}_{TR}$ and $\mathcal{M}_R$

Properties of code shortening

$$\forall A \in \mathbb{F}_q^{n\times n}: \quad \begin{aligned} \|A\|_{TR} - s \le \|\sigma_{\mathcal{J}}^{(s)}(A)\|_{TR} \le \|A\|_{TR}, \\ \|A\|_{R} - s \le \|\sigma_{\mathcal{J}}^{(s)}(A)\|_{R} \le \|A\|_{R}, \end{aligned}$$

Codes via $p(x)$-circulants

Lemma

Let $\mathcal{C} = \sigma_{\mathcal{J}}^{(s)}(\mathcal{C}_{p(x)})$. Then

- $D_{TR}(\mathcal{C}) \geq D_{TR}(\mathcal{C}_{p(x)}) - s$
- $D_R(\mathcal{C}) \geq D_R(\mathcal{C}_{p(x)}) - s.$

Moreover, if $\mathcal{C}_{p(x)}$ is optimal in $\mathcal{M}_{TR}$ ($\mathcal{M}_R$) then $\mathcal{C}$ still be optimal in $\mathcal{M}_{TR}$ (resp. $\mathcal{M}_R$).

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Optimal codes in $\mathcal{M}_R$

Proposition

The code $\mathcal{C}_{p(x)}$ is optimal in $\mathcal{M}_R$ (i.e. $D_R(\mathcal{C}_{p(x)}) = n$) iff $p(x)$ is irreducible in $\mathbb{F}_q[x]$.

Corollary

The shortened code $\sigma_{\mathcal{J}}^{(n-m)}(\mathcal{C}_{p(x)})$ is optimal in $\mathcal{M}_R$ (i.e. $D_R(\mathcal{C}_{p(x)}) = m$) if $p(x)$ is irreducible in $\mathbb{F}_q[x]$.

Outline

Optimal codes in $\mathcal{M}_{TR}$

Theorem

The code $\mathcal{C}_{p(x)}$ is optimal in $\mathcal{M}_{TR}$ (i.e. $D_{TR}(\mathcal{C}_{p(x)}) = n$) when

(i) $p(x)$ *is an irreducible in* $\mathbb{F}_q[x]$;

(ii) $p(x) = x^n - p_0,\ p_0 \in \mathbb{F}_q^*$;

(iii) $p(x) = x^n - p_t x^t - p_0,\ t \in [1, n-1],\ p_0, p_t \in \mathbb{F}_q^*$.

Proof of the Theorem

Theorem (recall)

$$\forall A \in \mathbb{F}_q^{m\times n}: \ \|A\|_{TR} = \max_{\tau} |\Delta_\tau(A)|,$$

where τ *runs overall injections from* $[0,\ m-1]$ *to* $[0,\ n-1]$ *and* $|\Delta_\tau(A)|$ *– a number of nonzero entries of* A *on the* Δ_τ.

Lemma

A linear $[m\cdot n,\ n]_q$*-code* $\mathcal{C}$ *is an optimal in* $\mathcal{M}_{TR}$ *iff for any* $A \in \mathcal{C}\setminus\{0\}$ *there exists a diagonal* Δ_τ *such that* $|\Delta_\tau(A)| = m$.

Proof of the Theorem

Proof of (ii)

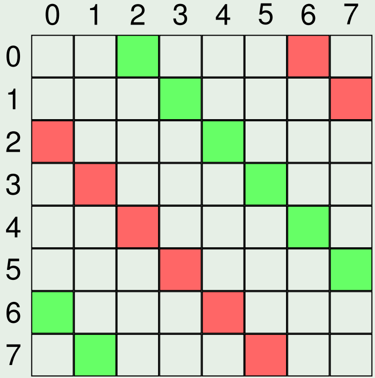
Let $A = \mu_{p(x)}(v)$, where $v = (a_0, \dots, a_{n-1})$ is the first row of A, be a nonzero $p(x)$-circulant. To simplify the notation for each $i \in [0, n-1]$ put

$$\Delta_i = \{(j, (i+j) \mod m) \mid j \in [0, m-1]\}.$$

In the case (ii) for some $i \in [0, n-1]$ we have $a_i \neq 0$ and the entries of A on the Δ_i are equal to a_i or $p_0 a_i$. So $|\Delta_i(A)| = n$ and (ii) is proved.

Proof of the Theorem

Example ($n = 8$, Δ_2 – green, Δ_6 – red)



Proof of the Theorem

Toward to the proof of (iii)

$\forall A \in \mathcal{C}_{p(x)}$, $p(x) = x^n - p_t x^t - p_0$, $t \in [1, n-1]$, $p_0, p_t \in \mathbb{F}_q^*$, there exists a bijection $\tau \in S_n$ such that

- $|\Delta_\tau(A)| = n$;
- $\exists j_1, j_2 \in [0,\ n-1] :\ \Delta_\tau \subseteq \Delta_{j_1} \cup \Delta_{j_2}$.

Optimal codes in $\mathcal{M}_{TR}$

Counterexample for $p(x) = x^n - p_t x^t - p_s x^s - p_0$, $0 < s < t < n$

Let $q = 2$, $p(x) = x^3 + x^2 + x + 1 = (x+1)^3$. Then

$$A = \mu_{p(x)}((1, 0, 1)) = \varphi(1 + x^2) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

has term rank 2.

Outline

Open questions

- More sufficient and necessary conditions on optimality of $\mathcal{C}_{p(x)}$ in $\mathcal{M}_{TR}$
- Decoding methods for $\mathcal{C}_{p(x)}$ in $\mathcal{M}_{TR}$
- Constructions for optimal codes with $2 < D_{TR} < m$