On the Hamming-Like Upper Bound on the Minimum Distance of LDPC Codes

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Outline

1. Task statement
2. Generalized LDPC codes
3. Irregular LDPC codes
4. Numerical results

Task statement

In

Y. Ben-Haim and S. Litsyn. *Upper Bounds on the Rate of LDPC Codes as a Function of Minimum Distance*. IEEE Trans. Inf. Theory, vol. 52, no. 5, pp. 2092–2100, May 2006.

a Hamming-like upper bound on the minimum distance of regular binary LDPC codes is given. We extend the bound to the case of irregular and generalized LDPC codes over $\mathbb{F}_q$.

Constituent code

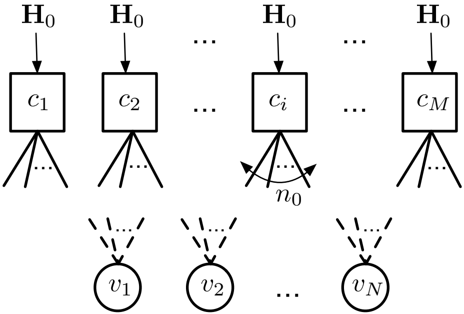
We assume $\mathcal{C}_0$ to be an $[n_0, R_0, d_0]$ code over $\mathbb{F}_q$. Let us denote the parity-check matrix of the *constituent code* by $\mathbf{H}_0$. The matrix has size $m_0 \times n_0$, where $m_0 = (1 - R_0)n_0$.

Let $G(s, n_0, d_0)$ be the *weight enumerator* of the code $\mathcal{C}_0$, i.e.

$$G(s, n_0, d_0) = 1 + \sum_{i=d_0}^{n_0} A(i)s^i,$$

where $A(i)$ is the number of codewords of weight i in a code $\mathcal{C}_0$.

Tanner graph



To check if $\mathbf{v} = (v_1, v_2, \ldots, v_N) \in \mathbb{F}_q^N$ is a codeword of $\mathcal{C}$ we associate the symbols of $\mathbf{v}$ to the variable nodes. The word $\mathbf{v}$ is called a codeword of $\mathcal{C}$ if all the constituent codes are satisfied.

Upper bound for generalized LDPC codes

Theorem

Let $\mathcal{C}$ be a generalized LDPC code of length N, rate R, minimum distance δN, with constituent $[n_0, R_0, d_0]$ code $\mathcal{C}_0$ over $\mathbb{F}_q$. Let $G(s, n_0, d_0)$ be the weight enumerator of $\mathcal{C}_0$. Then for sufficiently large N the following inequality holds

$$R \leq 1 - \frac{h_q(\delta/2)}{h_{q^{m_0}}\left[1 - (1-\delta/2)^{n_0} G\left(\frac{\delta/2}{(1-\delta/2)(q-1)}\right)\right]} + o(1),$$

where

$$h_Q(x) = -x \log_Q x - (1-x)\log_Q(1-x) + x \log_Q(Q-1).$$

is Q-ary entropy function.

Sketch of the proof

Consider all the possible vectors of length N, weight $W = \omega N$ over $\mathbb{F}_q$. We introduce an equiprobable distribution on such vectors. Let us consider the i-th check, by $\mathbf{S} = (\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_M)$ we denote the resulting syndrome of generalized LDPC code.

$$
\begin{aligned}
p_0 &= \Pr(\mathbf{S}_i = \mathbf{0}) \\
&= \frac{1}{\binom{N}{W}(q-1)^W} \left[\sum_{i=0}^{n_0} \left\{ A(i) \binom{N-n_0}{W-i} (q-1)^{W-i} \right\} \right].
\end{aligned}
$$

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\end{aligned}
$$

In what follows we are interesting in asymptotic estimate when $N \to \infty$. In this case we have

$$
\begin{aligned}
p_0 &= \left[\sum_{i=0}^{n_0} \left\{ A(i) \omega^i (1-\omega)^{n_0-i} (q-1)^{-i} \right\} \right] + o(1) \\
&= (1-\omega)^{n_0} G\left(\frac{\omega}{(1-\omega)(q-1)}, n_0, d_0 \right) + o(1).
\end{aligned}
$$

Sketch of the proof

Let $H(X)$ be the *binary* entropy of the random variable X,

By the log-sum inequality.

$$\begin{aligned} H(\mathbf{S}_i) &= -\sum_{j=0}^{q^{m_0}-1} \Pr(\mathbf{S}_i = j) \log_2 \Pr(\mathbf{S}_i = j) \\ &\leq -p_0 \log_2 p_0 - (1-p_0) \log_2 \frac{1-p_0}{q^{m_0}-1} \\ &= h_{q^{m_0}}(1-p_0) \log_2 q^{m_0}. \end{aligned}$$

Sketch of the proof

For $\omega < \delta/2$

$$\frac{1}{N} H(\mathbf{S}) = h_q(\omega) \log_2 q + o(1),$$

as all the syndromes corresponding to such vectors are different.

$$H(\mathbf{S}) \leq \sum_{i=1}^{M} H(\mathbf{S}_i) = M h_{q^{m_0}}(1 - p_0) \log_2 q^{m_0}.$$

Notation

The constituent code in this case is a single parity-check (SPC) code over $\mathbb{F}_q$. Thus the enumerator of an SPC code over $\mathbb{F}_q$ is as follows

$$G(s, d_0 = 2, n_0) = \frac{1}{q}(1 + (q-1)s)^{n_0} + \frac{q-1}{q}(1-s)^{n_0}.$$

To formulate a theorem we need a notion of row degree polynomial

$$\rho(x) = \sum_{i=r_{\min}}^{r_{\max}} \rho_i x^i,$$

where ρ_i is a fraction of rows of the parity check matrix of weight i, $r_{\min}$ and $r_{\max}$ are the minimal and maximal row weights accordingly.

Upper bound for irregular LDPC codes

Theorem

Let $\mathcal{C}$ be an LDPC code of length N, rate R, minimum distance δN, with row degree polynomial $\rho(x)$. Then for sufficiently large N the following inequality holds

$$R \leq \overline{R}(q, \rho(x)) = 1 - \frac{h_q(\delta/2)}{h_q\left[\frac{q-1}{q}\left(1 - \rho\left(1 - \frac{q}{q-1}\delta/2\right)\right)\right]} + o(1).$$

Proof

$$
\begin{aligned}
&\frac{1}{\log_2 q}\sum_{i=1}^{M} H(\mathbf{S}_i)\\
&= (1-R)\sum_{i=r_{\min}}^{r_{\max}} \rho_i h_q\left[1-(1-\omega)^{n_0} G\left(\frac{\omega}{(1-\omega)(q-1)}\right)\right]\\
&= (1-R)\sum_{i=r_{\min}}^{r_{\max}} \rho_i h_q\left[\frac{q-1}{q}-\frac{q-1}{q}\left(1-\frac{q}{q-1}\omega\right)^i\right]\\
&\le (1-R)h_q\left[\frac{q-1}{q}-\frac{q-1}{q}\rho\left(1-\frac{q}{q-1}\omega\right)\right].
\end{aligned}
$$

Analysis

Proposition

Let $\ell > 0$ be an integer, let $\rho(x)$ be the row degree distribution of irregular code, such that $\sum_{i=r_{\min}}^{r_{\max}} i\rho_i = \ell$ and let $\rho_{reg} = x^\ell$, then

$$\overline{R}(q, \rho(x)) \leq \overline{R}(q, \rho_{reg}(x)).$$

Numerical results for $q = 8$

As an example we choose regular $(\ell = 3, n_0)$ LDPC codes. We see that at very high rates $(R > 0.994)$ the bound lies below the Varshamov–Gilbert bound. We note that the interval of rates in which we observe this behavior is decreasing when q grows. For $q = 2$ the interval is $R > 0.985$, for $q = 16$ the interval is $R > 0.997$.

(ℓ, n_0); R	(3,10); 0.7	(3,50); 0.94	(3,100); 0.97	(3,200); 0.985	(3,500); 0.994	(3,600); 0.995
VG	0.1260	0.0179	0.0080	0.0036	**0.0013**	**0.0011**
New	0.2282	0.0263	0.0106	0.0043	**0.0013**	**0.0010**
PL	0.2625	0.0525	0.0262	0.0131	0.0052	0.0044
BE	0.2338	0.0355	0.0160	0.0073	0.0026	0.0021
MRRW	0.2494	0.0545	0.0281	0.0144	0.0059	0.0050

Thank you for the attention!