Multiaccess problem with two active users.

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3. New algorithm w(N,2,t)

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2. Group testing with $\mathcal{D} = 2$
3. New algorithm w(N,2,t)
4. Main result: for $N = \lfloor 2^{\frac{t+1}{2}} - t2^{\frac{t}{4}} \rfloor$ the problem can be solved in t tests

Classical group testing

- $[N] := \{1, \dots, N\}$ the set of elements
- $\mathcal{D} \subset [N]$ the set of defective elements

The main problem of group testing is determining $\mathcal{D}$ in the fewest number of tests. Each test is some subset of $[N]$. It is assumed that there is a test function which for any subset $\mathcal{S} \subset [N]$ indicates the presence of a defective in this subset (gives an answer to the test). Formally, a test function $f_{\mathcal{S}} : 2^{[N]} \to \{0, 1\}$ can be defined as follows:

$$f_{\mathcal{S}}(\mathcal{D}) = \begin{cases} 0 & \text{if} \quad |\mathcal{S} \cap \mathcal{D}| = 0 \\ 1 & \text{if} \quad |\mathcal{S} \cap \mathcal{D}| > 0. \end{cases} \tag{1}$$

A set of tests forms a search algorithm. We say that a search algorithm is successful if after applying it we can uniquely determine $\mathcal{D}$ from the answers $f_{\mathcal{S}_1}, \dots, f_{\mathcal{S}_t}$.

Algorithms can be adaptive and nonadaptive. In an adaptive algorithm, when choosing a test one can use results of previous tests. In a nonadaptive algorithm, all tests are independent.

We consider only adaptive strategies and worst case analysis.

Let $|\mathcal{D}| = D$ be the number of defectives, and $N_t(D)$ the largest number of elements among which D defectives can be found in t tests. For an adaptive algorithm $a = a(N, D, t)$, denote by $a_t(D)$ the maximal number of elements for which it is proved that the $D, a_t(D)$ problem can be solved in t tests, i.e., algorithm a is successful. Thus, $a_t(D)$ is a lower bound for $N_t(D)$.

Group testing with $\mathcal{D} = 2$

In 1982, G. J. Chang, F. K. Hwang , and S. Lin obtained an upper bound $N_t(2) \leq \lfloor 2^{(t+1)/2} - 1/2 \rfloor$ and proved for a search algorithm I proposed by them that $\frac{I_t(2)}{N_t(2)} > 0.95$.

t	2	3	4	5	6	7	8	9	10	11
$N_t(2) =$	3	4	5	7	10	15	22	31	44	63

Example. $N_6(2) = 10$
If we take 11 then we have $\binom{11}{2} = 55 < 2^6$. But after first test (we must ask more then 4 elements) we have

$$\binom{4}{2} + 4 \cdot 7 = 34 > 2^5$$

This result was improved in [3], where a proposed search algorithm u yielded the bound $\frac{u_t(2)}{N_t(2)} > 0.983$. Moreover, it was also shown in that for a search algorithm v there exists t_0, such that $\frac{v_t(2)}{N_t(2)} > 0.995$ for $t \geq t_0$.

t	12	13	14	15	16	17	18	19	20
$N_t(2) \leqslant$	90	127	180	255	361	511	723	1023	1447
$N_t(2) \geqslant$	89	126	178	252	357	506	717	1015	1437

So there was constructed an algorithm solving the $(2, u_t)$ problem in t tests, where

$$u_t = \begin{cases} 89 \cdot 2^{k-6} & \text{for } t = 2k \geq 12, \\ 63 \cdot 2^{k-5} & \text{for } t = 2k+1 \geq 13. \end{cases}$$

To conclude this section, we present a result which will be used below. It was obtained in for a special case where two defectives are contained in two disjoint subsets of $[N]$, one in each subset.

Lemma

Assume that a set $A \subset [N]$ is known to contain exactly one defective, a set $B \subset [N]$ is also known to contain exactly one defective, and these sets are disjoint. Then the minimal number of tests required to find the defective in A, $|A| = m$, and the defective in B, $|B| = n$, is $\lceil \log mn \rceil$.

So we have the upper bound

$$N_t(2) \leq \lfloor 2^{(t+1)/2} - 1/2 \rfloor$$

and for $N = C \cdot 2^{\frac{t+1}{2}}$ elements the problem can be solved in t tests.

New algorithm $w(N, 2, t)$

Consider $w_t = \lfloor 2^{\frac{t+1}{2}} - t2^{\frac{t}{4}} \rfloor$ elements, and let $[N] = \mathcal{Z} \cup \mathcal{X} \cup \mathcal{Y}$ be a partition for which we may claim that $f_{\mathcal{Z}}(\mathcal{D}) = 0$, $f_{\mathcal{X}}(\mathcal{D}) = 1$.

- **Test 1:** For the first test, take $\mathcal{S}_1 = [1, x_1]$, where $x_1 = \lfloor (\sqrt{2} - 1)2^{\frac{t}{2}} \rfloor$.

$$A_1 = \binom{x_1}{2} + x_1 \cdot (w_t - x_1) \leq 2^{t-1} - t(\sqrt{2} - 1)2^{\frac{3t}{4}}$$

- **Test 2 (after the answer $f_{S_1} = 1$):**
  For the second test, take $\mathcal{S}_2 = [1, x_2]$, where x_2 is an integer such that the number

$$A_2 = \binom{x_2}{2} + x_2 \cdot (w_t - x_2)$$

  is the nearest to $A_1/2$.

New algorithm $w(N, 2, t)$

After that, we proceed similarly. Assume that after k tests we obtain a partition $\mathcal{Z} = [1, z_k]$, $\mathcal{X} = [z_k + 1, z_k + x_k]$, $\mathcal{Y} = [z_k + x_k + 1, w_t]$.

- **Test** $(k + 1)$**:** For the $(k + 1)$st test, we take the interval $[z_k + 1, z_k + x_{k+1}]$ of length x_{k+1}, where x_{k+1} is an integer such that

$$A_{k+1} = \binom{x_{k+1}}{2} + x_{k+1} \cdot y_{k+1} \qquad (2)$$

  is the nearest to $A_k/2$.

- **Test** $(k + 2)$**:** For the $(k + 2)$nd test, take the first $\lfloor \frac{2^{t-k-2}}{x_{k+1}} \rfloor$ elements of $\mathcal{Y}$ (if there are less elements in $\mathcal{Y}$, take all of them).
- **Test** $(2k + 1)$**:** Take $\lfloor \frac{2^{t-2k-1}}{x_{k+1}} \rfloor$ elements of $\mathcal{Y}$ (if there are less elements remaining in $\mathcal{Y}$, take all of them).

Main result

Theorem

For the adaptive algorithm $w = w(N, D, t)$ we have

$$w_t(2) = \lfloor 2^{\frac{t+1}{2}} - t2^{\frac{t}{4}} \rfloor.$$

Corollary. *As $t \to \infty$, we have*

$$\frac{w_t}{N_t(2)} \to 1.$$

Now we show that using the idea of our algorithm $w(N, 2, t)$ we obtain good results also for some small N.

Theorem

We have

$$N_{20}(2) \geq 1438.$$

$$\mathcal{S}_1 = [1, 423]$$

$$\mathcal{S}_2 = [1, 193]$$

- after the answer $f_{S_2} = 1$ we ask 679, 339 and 169. Then we have 251 elements and we can find 2 defectives for 15 tests.
- after the answer $f_{S_2} = 0$ we ask 569, 284 and 142. Then we have 250 elements and we can find 2 defectives for 15 tests.

References

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Thank you for your attention!

Assume that there are N users who transmit their messages in the form of binary sequences of length t.

During transmission through a channel, exactly two users can be active (i.e., transmit their messages according to a transmission strategy designed beforehand).

At each time instant (from 1 to t), the channel output is 0 if both users transmit zeros at this instant, and in all other cases the channel output is 1.

A transmission strategy must be such that, given the channel output, one can find out which users transmitted their messages.