On nested completely regular codes and distance regular graphs

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Outline

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In the usual way, i.e., as coset graphs, infinite families of embedded distance-regular coset graphs of diameter $D = 3$ and 4 are constructed.
In some cases, the constructed codes are also completely transitive codes and the corresponding coset graphs are distance-transitive.

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Two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_2^n$ are *neighbors* if $d(\boldsymbol{x}, \boldsymbol{y}) = 1$.
For a binary code C with covering radius ρ define

$$C(i) \;=\; \{\boldsymbol{x} \in \mathbb{F}_2^n : \; d(\boldsymbol{x}, C) = i\}, \;\; i = 1, 2, \ldots, \rho.$$

Definition 1.

A code C with covering radius ρ is completely regular, if for all $l \geq 0$ every vector $x \in C(l)$ has the same number c_l of neighbors in $C(l-1)$ and the same number b_l of neighbors in $C(l+1)$.

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Define $a_l = n - b_l - c_l$ and note that $c_0 = b_\rho = 0$.
Also define $(b_0, \dots, b_{\rho-1}; c_1, \dots, c_\rho)$ as the intersection array of C.

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$(Sole[1990])$ A binary linear code C with covering radius ρ is completely transitive if $\mathrm{Aut}(C)$ has $\rho + 1$ orbits when acts on the cosets of C.

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Clearly any completely transitive code is completely regular.

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Let Γ be a finite connected simple graph (i.e., undirected, without loops and multiple edges).

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An *automorphism* of a graph Γ is a permutation π of the vertex set of Γ such that, for all $\gamma, \delta \in \Gamma$ we have $d(\gamma, \delta) = 1$, if and only if $d(\pi\gamma, \pi\delta) = 1$.

Definition 3.

$(Brouwer, Cohen, Neumaier[1989])$ A simple connected graph Γ is called *distance-regular* if it is regular of valency k, and if for any two vertices $\gamma, \delta \in \Gamma$ at distance i apart, there are precisely c_i neighbors of δ in $\Gamma_{i-1}(\gamma)$ and b_i neighbors of δ in $\Gamma_{i+1}(\gamma)$.

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The sequence $(b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D)$, where D is the diameter of Γ, is called the *intersection array* of Γ. Clearly $b_0 = k, \quad b_D = c_0 = 0, \quad c_1 = 1.$

Let C be a linear completely regular code with covering radius ρ and intersection array $(b_0, \ldots, b_{\rho-1}; c_1, \ldots c_\rho)$. Let $\{B\}$ be the set of cosets of C.

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Define the graph Γ_C, which is called the *coset graph of* C, taking all different cosets $B = C + \mathbf{x}$ as vertices, with two vertices $\gamma = \gamma(B)$ and $\gamma' = \gamma(B')$ adjacent, if and only if the cosets B and B' contain neighbor vectors, i.e., there are $\mathbf{v} \in B$ and $\mathbf{v}' \in B'$ such that $d(\mathbf{v}, \mathbf{v}') = 1$.

Lemma 4.

$(Brouwer, Cohen, Neumaier[1989], Rifa, Pujol[1991])$ *Let C be a linear completely regular code with covering radius ρ and intersection array $(b_0, \ldots, b_{\rho-1}; c_1, \ldots c_\rho)$ and let Γ_C be the coset graph of C.*

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Then Γ_C is distance-regular of diameter $D = \rho$ with the same intersection array as the code C.
If C is completely transitive, then Γ_C is distance-transitive.

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P_m is the vertical join of H_m and E_m.
The code $C^{(u)}$ with parity check matrix P_m is a cyclic binary completely regular $[n, n-(m+u), 3]$ code with $\rho = 3$ and with generator polynomial $g(x) = m_\alpha(x)m_{\alpha^r}(x) \in \mathbb{F}_2[x]$, where $m_{\alpha^i}(x)$ means the minimal polynomial of α^i (Calderbank, Goethals, 1984).

Nested antipodal distance regular graphs

For $i \in \{0, \ldots, u\}$, taking $u - i$ cosets $C^{(u)} + \boldsymbol{v}_1, \ldots, C^{(u)} + \boldsymbol{v}_{u-i}$ with linearly independent syndromes $S(\boldsymbol{v}_1), \ldots, S(\boldsymbol{v}_{u-i})$, we generate a linear binary code $C^{(i)} = \langle C^{(u)}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{u-i} \rangle$.

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All the constructed codes contains $C^{(u)}$ and they are contained in the Hamming code $C^{(0)}$.

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We conjecture that codes $C^{(i)}$ and $C^{(i)*}$ are completely transitive if and only if $i = 0, i = 1, i = u$ or $2^i \leq u + 1$, for $i \in \{2, \dots, u - 1\}$.

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From Theorem 5 and Lemma 7 we have

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For any $m = 2u \geq 4$ and $i = 0, 1, \ldots, u$ there exist a family of embedded antipodal distance-regular coset graphs $\Gamma^{(i)*}$ with 2^{m+i+1} vertices and diameter 4.

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For any $m = 2u \geq 4$ *and* $i = 0, 1, \ldots, u$ *there exist a family of embedded antipodal distance-regular coset graphs* $\Gamma^{(i)*}$ *with* 2^{m+i+1} *vertices and diameter* 4. *Specifically:*

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(ii) $\Gamma^{(i)*}$ *is a subgraph of* $\Gamma^{(i+1)*}$ *for all* $i = 0, 1, \ldots, u - 1$.
(iii) $\Gamma^{(i)*}$ *covers* $\Gamma^{(j)*}$, *where* $j = 0, 1, \ldots, i - 1$ *with the size of the fibre* $r_{i,j} = 2^{i-j}$.
(iv) $\Gamma^{(i)*}$ *is distance-transitive for* $i = 0, 1, u$ *when* $m \geq 8$ *and* $i = 0, 1, 2, 3$ *when* $m = 6$.

Theorem 9.

For any $m = 2u \geq 4$ *and* $i = 0, 1, \ldots, u$ *there exist a family of embedded antipodal distance-regular coset graphs* $\Gamma^{(i)*}$ *with* 2^{m+i+1} *vertices and diameter* 4. *Specifically:*
(i) $\Gamma^{(i)*}$ *has intersection array* $(2^m, 2^m - 1, 2^m - 2^{m-i}, 1; 1, 2^{m-i}, 2^m - 1, 2^m)$.
(ii) $\Gamma^{(i)*}$ *is a subgraph of* $\Gamma^{(i+1)*}$ *for all* $i = 0, 1, \ldots, u-1$.
(iii) $\Gamma^{(i)*}$ *covers* $\Gamma^{(j)*}$, *where* $j = 0, 1, \ldots, i-1$ *with the size of the fibre* $r_{i,j} = 2^{i-j}$.
(iv) $\Gamma^{(i)*}$ *is distance-transitive for* $i = 0, 1, u$ *when* $m \geq 8$ *and* $i = 0, 1, 2, 3$ *when* $m = 6$.

The first graphs $\Gamma^{(1)}$ and $\Gamma^{(1)*}$ are well known distance-transitive graphs (Borges, Rifa, Zinoviev, 2014). Graphs $\Gamma^{(u)}$ and $\Gamma^{(u)*}$ are also known (Calderbank, Goethals, 1984; Brouwer, Cohen, Neumaier, 1989). All graphs $\Gamma^{(i)}$ for $i = 0, 1, \ldots, u$ have been constructed by Godsil and Hensel (1992) using the Quotient Construction. But it was not mentioned in all references above that some of these graphs are completely transitive. Besides, except for the graphs $\Gamma^{(u)}$, it was not stated that these graphs can be constructed as coset graphs. The graphs $\Gamma^{(i)*}$ for $i = 2, \ldots, u-1$ seems to be new.