HOMOGENEOUS ARCS IN PROJECTIVE HJELMALEV SPACES

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Chain Rings

Definition. A ring (associative, $1 \neq 0$, ring homomorphisms preserving 1) is called a **left (right) chain ring** if the lattice of its left (right) ideals forms a chain.

A. Nechaev, Mat. Sbornik **20**(1973).

$$R > \mathrm{rad}\, R > (\mathrm{rad}\, R)^2 > \ldots > (\mathrm{rad}\, R)^{m-1} > (\mathrm{rad}\, R)^m = (0).$$

- m – the length of R;
- $\mathbb{F}_q$ – the residue field of R, $q = p^s$;
- p^r – the characteristic of R.

Example. (Chain Rings with q^2 Elements)

$$R\colon |R| = q^2,\ R/\operatorname{rad} R \cong \mathbb{F}_q;\quad R > \operatorname{rad} R > (0).$$

R. Raghavendran, Compositio Mathematica **21** (1969).

A.Cronheim, Geom. Dedicata **7**(1978).

If $q = p^s$ there exist $s+1$ isomorphism classes of such rings:

- σ-dual numbers over $\mathbb{F}_q$, $\forall \sigma \in \operatorname{Aut} \mathbb{F}_q$: $R_\sigma = \mathbb{F}_q[X;\sigma]/(X^2)$.

  $(a_0 + a_1 X)(b_0 + b_1 X) = a_0 b_0 + (a_0 b_1 + a_1 \sigma(b_0))X$.

  Also: $\mathbb{S}_q^{(i)} = \mathbb{F}_q[X;\phi^i]/(X^2)$, where $\phi : \alpha \to \alpha^p$.

- the Galois ring $\mathrm{GR}(q^2, p^2) = \mathbb{Z}_{p^2}[X]/\,(f(X))$, $f(X)$ is monic of degree r, irreducible mod p.

  Also: $\mathbb{T}_q = \mathrm{GR}(q^2, p^2)$.

The Homogeneous Weight

Definition. Let R be a finite ring. A mapping $w\colon R \to \mathbb{R}$ is called a a normalized homogenious weight if $(x, y) \mapsto w(x - y)$ is a metric on R and if the following two axioms hold:

(1) $w(x) = w(uxv)$ for all $x \in R$ and $u, v \in R^\times$.

(2) the average weight $\frac{1}{|I|}\sum_{x\in I} w(x)$ for every left or right ideal $I \neq \{0\}$ of R is equal to one.

Theorem. (I. Constantinescu, W. Heise, 1997) A homogenious weight on the integer residue ring $\mathbb{Z}_m$ exists if and only if m is not divisible by 6. In such case the normalized homogenious weight is unique.

The Heise-weight: the normalized homogeneous weight on $R = \mathbb{Z}_m$:

$$w_R(x) = 1 - \frac{\mu(m/d)}{\varphi(m/d)},$$

where $d = gcd(x, m)$.

Let R be a local ring with residue field of order q. Then for all $x \in R$:

$$w_R(x) = \begin{cases} 1 & \text{if } x \notin \operatorname{soc}(R), \\ \frac{q}{q-1} & \text{if } x \in \operatorname{soc}(R) \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

For chain rings R with $|R| = q^m$, $R/\operatorname{rad} R \cong \mathbb{F}_q$:

$$w_R(x) = \begin{cases} 1 & \text{if } x \in R \setminus (\operatorname{rad} R)^{m-1}, \\ \frac{q}{q-1} & \text{if } x \in (\operatorname{rad} R)^{m-1} \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

Let R be a chain ring with $|R| = q^2$, $R/\operatorname{rad} R \cong \mathbb{F}_q$.

Definition. A code over $\mathbb{F}_q$ is said to be linearly representable over R if it is the image of an R-linear code under the Reed-Solomon map:

$$\psi_{\mathsf{RS}} : \begin{cases} R & \to & \mathbb{F}_q^q \\ r = r_0 + r_1\theta & \to & (r_0, r_1) \begin{pmatrix} 0 & 1 & \zeta & \dots & \zeta^{q-2} \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}. \end{cases}$$

Here ζ is a primitive element of $\mathbb{F}_q$, and $r_i \in \Gamma$, where Γ is a set of elements from R no two of which are congruent modulo $\operatorname{rad} R$.

$$\psi_{\mathsf{RS}} : (R, w_{\mathsf{hom}}) \longrightarrow (\mathbb{F}_q^q, \frac{1}{q-1} w_{\mathsf{Ham}}).$$

The Projective Hjelmslev Geometries $\mathrm{PHG}({}_RR^n)$

- $M = {}_RR^n$;
- $\mathcal{P} = \{xR \mid x \in M \setminus M\theta\}$;
- $\mathcal{L} = \{xR + yR \mid x, y \text{ linearly independent}\}$;
- $I \subseteq \mathcal{P} \times \mathcal{L}$ – incidence relation;
- $\frown$ - **neighbour relation**:

  (N1) $X \frown Y$ if $\exists s, t \in \mathcal{L}$: $X, Y\, I\, s, X, Y\, I\, t$;

  (N2) $s \frown t$ if $\forall X\ I\ s\ \exists Y\ I\ t$: $X \frown Y$ and $\forall Y\ I\ t\ \exists X\ I\ s$: $Y \frown X$.

Definition. The incidence structure $\Pi = (\mathcal{P}, \mathcal{L}, I)$ with neighbour relation $\bigcirc$ is called the (**left**) **projective Hjelmslev geometry** over the chain ring R.

Notation: $\mathrm{PHG}({}_R R^k)$, $\mathrm{PHG}(k-1, R)$

$\mathcal{P}'$ – the set of all neighbour classes on points

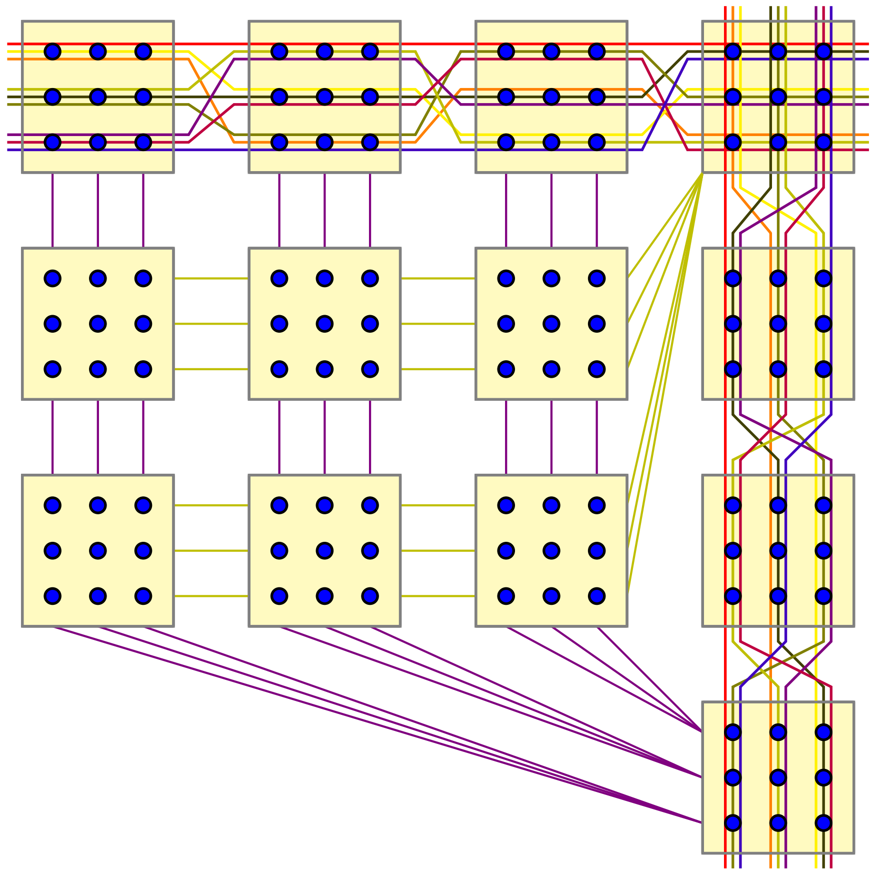
$\mathcal{L}'$ – the set of all neighbour classes on lines

$I' \subseteq \mathcal{P}' \times \mathcal{L}'$ – incidence relation defined by

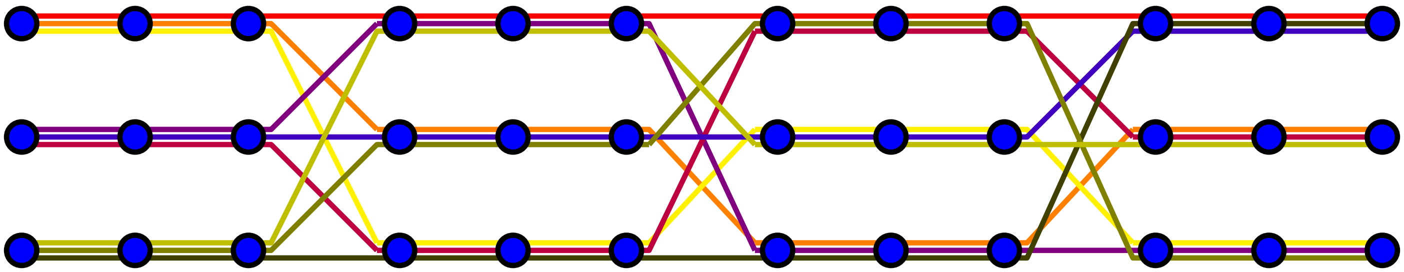
$$[P]I'[l] \Leftrightarrow \exists P_0 \in [P], \exists l_0 \in [l], P_0 I l_0.$$

Theorem. $(\mathcal{P}', \mathcal{L}', I') \cong \mathrm{PG}(k-1, q)$.

$\mathrm{PHG}(\mathbb{Z}_9^3)$



A Neighbour class of lines in $\mathrm{PHG}(\mathbb{Z}_9^3)$



Homogeneous Arcs in $\mathrm{PHG}({}_RR^n)$

For any subspace $S \subset \mathcal{P}$ define the homogenious weight of S by:

$$w(S) = \mathcal{K}(S) - \frac{1}{q-1}\mathcal{K}([S] \setminus S).$$

Definition. The mapping $\mathcal{K} : \mathcal{P} \to \mathbb{N}_0$ is called a homogeneous (N, W)-arc if

(a) $\mathcal{K}(\mathcal{P}) = N$;

(b) $w(H) \leq W$ for any hyperplane;

(c) $w(H_0) = W$ for at least one hyperplane H_0.

Definition. The mapping $\mathcal{K} : \mathcal{P} \longrightarrow \mathbb{N}_0$ is called a homogeneous (N, W)-blocking set if

(a) $\mathcal{K}(\mathcal{P}) = N$;

(b) $w(H) \geq W$ for any hyperplane;

(c) $w(H_0) = W$ for at least one hyperplane H_0.

Theorem. A linearly representable q-ary code with parameters

$$\big(qN, q^{2k}, (q-1)(N-W)\big)$$

exists if and only if there exists a homogeneous (N, W)-arc in $\mathrm{PHG}({}_R R^k)$.

Theorem. Let $\mathcal{K}$ be a projective arc in Π with homogeneous weights: $w_1 \leq w_2 \leq \ldots \leq w_s$. Then the complementary arc $\overline{\mathcal{K}} := 1 - \mathcal{K}$ has homogeneous weights $-w_s \leq w_{s-1} \leq \ldots \leq -w_1$.

Corollary. If $\mathcal{K}$ is a (N, W)-arc then $\overline{\mathcal{K}}$ is a $(N, -W)$-blocking set.

Homogeneous Arcs with $W = 0$

Theorem. Let $\mathcal{K}$ be a (N, W)-arc in $\mathrm{PHG}({}_RR^k)$. Then

$$\sum_H w(H) = 0.$$

Theorem. A homogeneous (N, W)-arc in $\mathrm{PHG}({}_RR^k)$ has $W = 0$ if and only if it is a (weighted) sum of neighbour classes of points.

Proof. Sufficiency – obvious.

Necessity. Depends on the rank of the point-by-hyperplanes incidence matrix. (L.-V., WCC, Bergen, 2013)

Corollary. A non-trivial homogeneous (N, W)-arc has $W \geq 1/q - 1$.

Families of Two-Weight Homogeneous Arcs

- s points from each point class of the projective line $\mathrm{PHG}(RR^2)$:
- the subgeometry $\mathrm{PG}(2,q)$ in $\mathrm{PHG}({}_RR^3)$, where $R = \mathbb{F}_q[X;\sigma]/(X^2)$;
- s parallel hyperplane segments with all possible directions in $\mathrm{PHG}({}_RR^k)$; q^2+q+1 line segments in $\mathrm{PHG}\ {}_RR^3)$ with all possible directions in the factor geometry;
- a hyperoval in $\mathrm{PHG}({}_RR3)$, where R is a chain ring of nilpotency index 2 and characteristic 4.
- a sporadic 39-arc in $\mathrm{PHG}(\mathbb{Z}_9^3)$.