FOURTEENTH INTERNATIONAL WORKSHOP ON ALGEBRAIC AND COMBINATORIAL CODING THEORY

PACKINGS OF SPHERICAL CAPS OF DIFFERENT RADII ON $\mathbb{S}^2$

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Problems

Problem 1. Given numbers $r_1, r_2, \ldots, r_n$ (not necessarily distinct) find the smallest radius of a sphere in $\mathbb{R}^3$ that can contain n non-intersecting spherical caps $SC_1, SC_2, \ldots, SC_n$ of radii $r_1, r_2, \ldots, r_n$, respectively.

Problem 1′ Given numbers $r_1, r_2, \ldots, r_n$ (not necessarily distinct) find the largest value of S/S', where S' is the area of a sphere in $\mathbb{R}^3$ that contains n non-intersecting spherical caps $SC_1, SC_2, \ldots, SC_n$ of radii $r_1, r_2, \ldots, r_n$, respectively, and S is the total area of these caps.

Problem 2. Compare (Riesz energy) against (Good packing) for the obtained configurations.

Some references

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Simulated-annealing-type algorithm

In general – analogous to other simulated annealing (SA) algorithms. It is a non-deterministic algorithm, based on random movements in the configuration.

Generating numerous (many thousands) different temporary configurations by gradually modifying the initial one

At each step decision to choose or not to choose new configuration, if new then not always selecting the best one.

The changes of the configurations are decided according to parameter which we call p-energy of the system. (counterpart of the thermal energy in the physical process of annealing). Good packings correspond to minimization of the p-energy.

General requirements

The major difficulties in such algorithms come from two general requirements:

(1) the initial temperature should be high enough

(2) the decreasing must be as slow as necessary to avoid the traps of the local extrema.

Different optimization problems require different approaches and adjustments during the implementation – no general rules.

General description

The main body of our algorithm includes four procedures (cycles):

Two of them (called outer) take care for decreasing the radius of the outer sphere and the decreasing of the temperature.

Other two cycles (called inner) perform measurements at each temperature step and try and perform moves inside the current configuration.

Local description

Number of the measurements (iterations) at each temperature step – dynamic.

At every iteration the movement cycle is performed, the new configuration is evaluated and accepted or rejected.

Four types of moves, called Exchange, Short, Leap and Neighbour, taken with certain probabilities.

Big-bang-type algorithm

The name suggests – this works (in some sense) opposite to SA.

The implementation starts with a small sphere to coincide with the largest cap to be packed. All other caps are placed arbitrarily on the initial sphere.

Two outer cycles increase the radius of the sphere and the temperature, respectively, in certain relations.

Two inner cycles take care for the iterations and moves, respectively. We prefer again dynamic numbers of the iterations and the moves. There are again four different moves – Swap (changes two caps), Short (moves some cap by short distance), Leap (like Short but by all possible angular distances) and Neighbour (exchanges two caps which are neighbours in certain sense).

Applications (1)

We applied both algorithms for many sets of radii for $n = 5$ and some other values of n. The final results coincide in almost all cases and usually give what one could expect as extremal configurations.

The densities vary from 81 to 93 per cent.

Note – our implementations of the BB algorithm appeared to be quite faster than these of the SA algorithm.

Applications (2)

We also analyzed the possible connections between the density of the obtained packings and their Riesz energy.

The Riesz energy of a spherical code $C \subset \mathbb{S}^{n-1}$ is defined by

$$W(C) = \sum_{x,y \in C, x \neq y} \frac{1}{[d(x,y)]^s},$$

where $s > 0$ is a real parameter. The classical problem asks for minimization of the energy for fixed dimension n, degree s and cardinality of the code C.

Applications (3)

Our calculations show that better packings have worse (i.e. larger) energy. For example, the best packing we found for spherical caps of radii sets $\{1, 1, 1, 1, 5\}$ and $\{1, 1, 1, 2, 5\}$ (after re-scaling, of course) have density approx. 92.8 and 91.1 per cent but the corresponding energies are approx. 17.16 and 15.46 (for $s = 2$), respectively. It is well-known open problem to find the minimum energy of a 5-point code on $\mathbb{S}^2$. The conjectured optimal code for some wide range of s is the by-pyramid with energy 8.50 for $s = 2$ (the best known lower bound in this case is 8.375).

Another possible comparison and analysis of our results come from the possibility to extend plane arrangements to $\mathbb{S}^2$ by using stereographic projections in different ways.

THANK YOU FOR YOUR ATTENTION !