INCREMENTAL CALCULATION OF decoding failure PROBABILITY FOR ITERATIVE DECODING OF a Reed-Solomon PRODUCT CODE.

Afanassiev Valentin B. and Davydov Alexander A.

IPPI RAS

Motivation

Products of Reed-Solomon codes are important in applications because they offer a combination of large blocks, low decoding complexity, and good performance [R. Blahut, J. Justesen, T. Hoeholdt and others…].

A real problem – calculate or estimate by simulation very low error probability for a given iterative decoding. It is known, that simulation is not applicable when expected error probability is less 10^{-8}

We give exact and approximate methods for calculation of an increment of failure probability for any iteration.

Product code construction and decoding algorithm

Product $(n \times m)$ code over $GF(q)$ with the component RS codes $\mathbb{C}_C(n,k,d)$ for columns and $\mathbb{C}_R(m,k,d)$ for rows of a matrix. Let $\mathbf{A}$ be a $k \times k$ message matrix then $\mathbf{C} = \mathbf{G}_C^T \mathbf{A} \mathbf{G}_R$ is a codeword of product code, where $\mathbf{G}_R, \mathbf{G}_C$ are generating matrixes of row and column component codes.

The product code has the minimal distance d^2 and has applicable decoding algorithm [R. Blahut] that corrects all configurations up to $(d^2 - 1)/2$ errors and much-much more.

Investigation of iterative decoding for product codes has a long history, nevertheless J. Justesen and T. Hoeholdt gave a new explanation of the iterative process with unavoidable degradation of the error model during iterative process.

We will follow this idea might be in more constructive way.

Supposition

The component code decoder works without decoding errors.

The decoding results can be:

correct decoding (with correction up to $t = \left(d - 1\right)/2$ errors) or

decoding failure (when more t errors).

That is not so extravagant supposition because probability of decoding error is always less $1/t!$.

A simplified version of product code decoder

L_R - list of undecoded rows and L_C list of undecoded columns.

At the beginning, all rows and columns are undecoded and error density equal to channel error probability.

During iteration, a component decoder examines undecoded blocks from the list L_C (or L_R) for correction $\leq t$ errors, when size of the list L_R (or L_C) is more $(d-1)$,

Other case, it corrects $\leq (d-1)$ erasures from the list of undecoded blocks, and deletes from the lists all successfully decoded blocks.

Remark 1: after the first iteration, the error density in all the rows and columns remaining in the lists L_C and L_R is higher than the channel error probability and higher than $\min\left[(t+1)/\left|L_C\right|,(t+1)/\left|L_R\right|\right]$.

Remark 2: the product code decoding failure condition: *size of the lists* L_C and L_R *have not changed during iteration (except the first one).* Additionally, detection of necessary correction of a position out of the lists L_C and L_R by any component decoder means detection of incorrect decoding of a block on some of earlier iteration.

Remark 3: de facto component decoders work on shortened component codes length of which is equal to size of the lists.

General scheme of calculation for iterative decoding

The probability of correct decoding of the product code after I iterations is

$$\Pr_C(I) = \sum_{i=1}^{I} \Delta_i ,$$

where Δ_i is an increment of the probability on i-th iteration.

Decoding failure probability (under accepted supposition) is just supplement of $\Pr_C(I)$ to one.

Let we start from column decoding.

Let ℓ_i be the number of undecoded "bad" blocks (size of L_C or L_R) after i-th iteration. Initial values are $|L_C| = \ell_0 = m, |L_R| = \ell_{-1} = n$.

After the first iteration, we get ℓ_1 undecoded "bad" columns.

If $\ell_1 < d$, the row component decoder corrects as erasures all m rows of the length ℓ_1. Other case the row decoder corrects all possible rows with $\leq t$ errors and defines the value ℓ_2 of undecoded "bad" rows.

Next, we have to recalculate estimate of the error density ρ_i in “bad” blocks taking into account their real (shortened) length.

Initial value - ρ_1 is the channel error probability.

After the first iteration, we define ρ_2 as ratio of average number of errors in “bad” columns to their height m. We will continue in this manner on the next iterations.

Exact formulas for calculation of the increment of failure probability

Definition 1. Error density in bad blocks on *i*-th iteration is

$$\rho_i = \frac{1}{\ell_{i-2}} \frac{\lambda^*\left(\ell_{i-2}, \rho_{i-1}\right)}{\lambda\left(\ell_{i-2}, \rho_{i-1}\right)},$$

$$\lambda^*\left(\ell, \rho\right) = \sum_{j=t+1}^{\ell} j \binom{\ell}{j} \rho^j \left(1-\rho\right)^{\ell-j},$$

$$\lambda\left(\ell, \rho\right) = \sum_{j=t+1}^{\ell} \binom{\ell}{j} \rho^j \left(1-\rho\right)^{\ell-j}$$

Definition 2. Probability of correct decoding of a block is

$$\gamma\left(\ell_{j-1}, \rho_j\right) = \sum_{v=0}^{t} \binom{\ell_{j-1}}{v} \left(\rho_j\right)^v (1-\rho_j)^{\ell-v} .$$

Definition 3. Probability of ℓ_j undecoded blocks conditioned to the state $\left(\ell_{j-1}, \ell_{j-2}\right)$ of last two iterations

$$\Pr\left(\ell_j \mid \ell_{j-1}, \ell_{j-2}\right) = \binom{\ell_{j-2}}{\ell_j} \left(\gamma\left(\ell_{j-1}, \rho_j\right)\right)^{\ell_{j-2}-\ell_j} \left(1-\gamma\left(\ell_{j-1}, \rho_j\right)\right)^{\ell_j} .$$

Definition 4. Increment of the probability of correct decoding on *i*-th iteration is as follows:

$$\Delta_i = \sum_{\ell_1=d+\delta(i,1)}^{n} \Pr\left(\ell_1 \mid m,n\right) \sum_{\ell_2=d+\delta(i,2)}^{m-1} \Pr\left(\ell_2 \mid \ell_1, m\right) \sum_{\ell_3=d+\delta(i,3)}^{\ell_1-1} \Pr\left(\ell_3 \mid \ell_2, \ell_1\right) \ldots \times$$

$$\times \sum_{\ell_{i-1}=d+\delta(i,i-1)}^{\ell_{i-3}-1} \Pr\left(\ell_{i-1} \mid \ell_{i-2}, \ell_{i-3}\right) \sum_{\ell_i=0}^{d-1} \Pr\left(\ell_i \mid \ell_{i-1}, \ell_{i-2}\right),$$

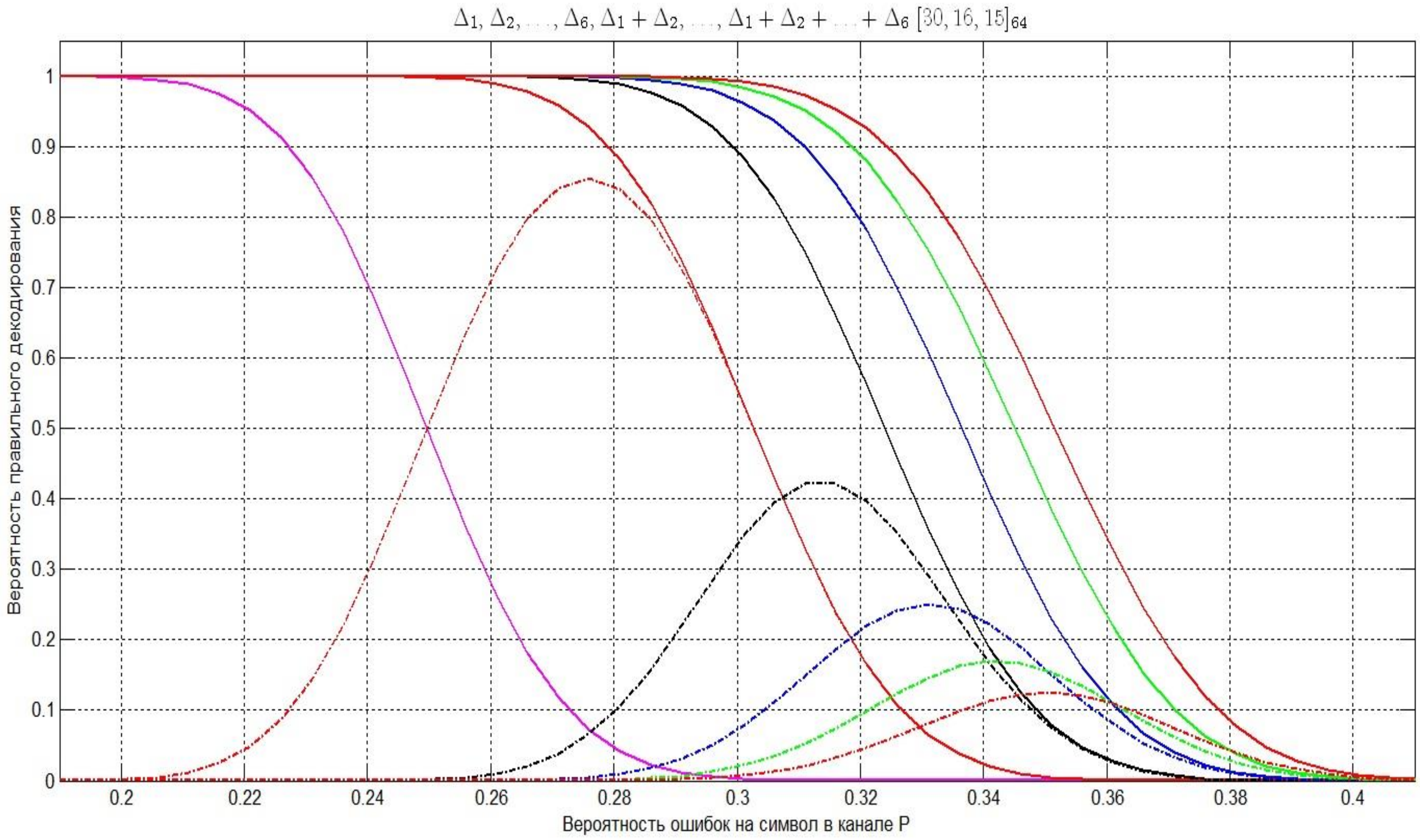
$$\delta(i,j) = \left\lfloor \frac{i-j-1}{2} \right\rfloor.$$

The upper and lower limits for summation defined in accordance with the rules of the product decoding failure.

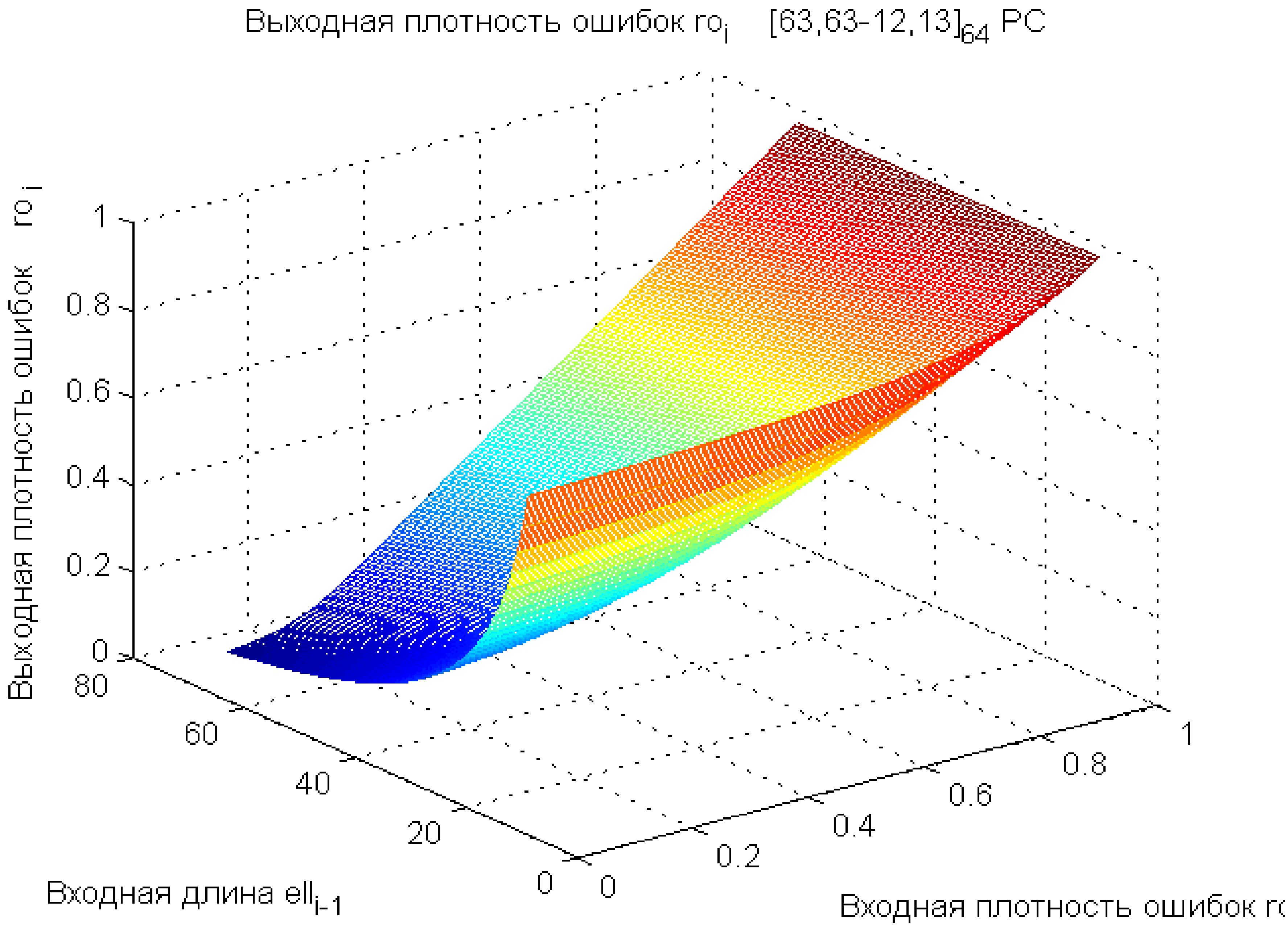
Complexity of calculation Δ_i is exponential with iteration number *i*.

Example of exact formulas for first iterations

	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7
Δ_1	$\sum_{\ell_1=0}^{d-1} \Pr(\ell_1)$						
Δ_2	$\sum_{\ell_1=d}^{n} \Pr(\ell_1)$	$\sum_{\ell_2=0}^{d-1} \Pr(\ell_2 \mid \ell_1)$					
Δ_3	$\sum_{\ell_1=d}^{n} \Pr(\ell_1)$	$\sum_{\ell_2=d}^{m-1} \Pr(\ell_2 \mid \ell_1)$	$\sum_{\ell_3=0}^{d-1} \Pr(\ell_3 \mid \ell_2 \ell_1)$				
Δ_4	$\sum_{\ell_1=d+1}^{n} \Pr(\ell_1)$	$\sum_{\ell_2=d}^{m-1} \Pr(\ell_2 \mid \ell_1)$	$\sum_{\ell_3=d}^{\ell_1-1} \Pr(\ell_3 \mid \ell_2 \ell_1)$	$\sum_{\ell_4=0}^{d-1} \Pr(\ell_4 \mid \ell_3 \ell_2)$			
Δ_5	$\sum_{\ell_1=d+1}^{n} \Pr(\ell_1)$	$\sum_{\ell_2=d+1}^{m-1} \Pr(\ell_2 \mid \ell_1)$	$\sum_{\ell_3=d}^{\ell_1-1} \Pr(\ell_3 \mid \ell_2 \ell_1)$	$\sum_{\ell_4=d}^{\ell_2-1} \Pr(\ell_4 \mid \ell_3 \ell_2)$	$\sum_{\ell_5=0}^{d-1} \Pr(\ell_5 \mid \ell_4 \ell_3)$		
Δ_6	$\sum_{\ell_1=d+2}^{n} \Pr(\ell_1)$	$\sum_{\ell_2=d+1}^{m-1} \Pr(\ell_2 \mid \ell_1)$	$\sum_{\ell_3=d+1}^{\ell_1-1} \Pr(\ell_3 \mid \ell_2 \ell_1)$	$\sum_{\ell_4=d}^{\ell_2-1} \Pr(\ell_4 \mid \ell_3 \ell_2)$	$\sum_{\ell_5=d}^{\ell_3-1} \Pr(\ell_5 \mid \ell_4 \ell_3)$	$\sum_{\ell_6=0}^{d-1} \Pr(\ell_6 \mid \ell_5 \ell_4)$	
Δ_7	$\sum_{\ell_1=d+2}^{n} \Pr(\ell_1)$	$\sum_{\ell_2=d+2}^{m-1} \Pr(\ell_2 \mid \ell_1)$	$\sum_{\ell_3=d+1}^{\ell_1-1} \Pr(\ell_3 \mid \ell_2 \ell_1)$	$\sum_{\ell_4=d+1}^{\ell_2-1} \Pr(\ell_4 \mid \ell_3 \ell_2)$	$\sum_{\ell_5=d}^{\ell_3-1} \Pr(\ell_5 \mid \ell_4 \ell_3)$	$\sum_{\ell_6=d}^{\ell_4-1} \Pr(\ell_6 \mid \ell_5 \ell_4)$	$\sum_{\ell_7=0}^{d-1} \Pr(\ell_7 \mid \ell_6 \ell_5)$



$\Delta_1, \Delta_2, \ldots, \Delta_6, \Delta_1 + \Delta_2, \ldots, \Delta_1 + \Delta_2 + \ldots + \Delta_6$ $[30, 16, 15]_{64}$
Вероятность правильного декодирования
Вероятность ошибок на символ в канале P
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.2
0.22
0.24
0.26
0.28
0.3
0.32
0.34
0.36
0.38
0.4


Выходная плотность ошибок ro_i [63,63-12,13]_64 РС
Выходная плотность ошибок ro_i
1
0.8
0.6
0.4
0.2
0
80
60
40
20
0
Входная длина ell_i-1
0
0.2
0.4
0.6
0.8
1
Входная плотность ошибок ro

Approximate recursive scheme for probabilities calculation

To decrease the complexity from exponential to a polynomial function let see on the last three terms in general expression for Δ_i:

$$\sum_{\ell_{i-2}=d+\delta(i,i-2)}^{\ell_{i-4}-1} \Pr\left(\ell_{i-2} \mid \ell_{i-3}, \ell_{i-4}\right),$$

$$\sum_{\ell_{i-1}=d+\delta(i,i-1)}^{\ell_{i-3}-1} \Pr\left(\ell_{i-1} \mid \ell_{i-2}, \ell_{i-3}\right), \qquad \sum_{\ell_i=0}^{d-1} \Pr\left(\ell_i \mid \ell_{i-1}, \ell_{i-2}\right).$$

These terms are related with a random shortened $\left(\ell_{i-1} \times \ell_{i-2}\right)$ subcode of the given product code. So we could find the probability $\Pr\left(\ell_{i-1}, \ell_{i-2}\right)$ and estimate the Δ_i for $i \geq 3$:

$$\Delta_i = \sum_{\ell_{i-2}=d}^{\tau(i-2)} \sum_{\ell_{i-1}=d}^{\tau(i-1)} \Pr^{\bullet}\left(\ell_{i-1}, \ell_{i-2}\right) \sum_{\ell_i=0}^{d-1} \Pr\left(\ell_i \mid \ell_{i-1}\ell_{i-2}\right),$$

$$\tau(i) = \ell_{-(i \bmod 2)} - \left\lfloor \frac{i}{2} \right\rfloor,$$

$$\Pr\left(\ell_i \mid \ell_{i-1}, \ell_{i-2}\right) = \binom{\ell_{i-2}}{\ell_i} \left(\gamma\left(\ell_{i-1}, \rho_i\right)\right)^{\ell_{i-2}-\ell_i} \left(1 - \gamma\left(\ell_{i-1}, \rho_i\right)\right)^{\ell_i},$$

$$\gamma\left(\ell_{i-1}, \rho_i\right) = \sum_{v=0}^{t} \binom{\ell_{i-1}}{v} \left(\rho_i\right)^v (1 - \rho_i)^{\ell_{i-1}-v},$$

Conditional density of errors

$$\rho_i\left(\ell_{i-2},\hat{\rho}_{i-1}\right)=\frac{1}{\ell_{i-2}}\frac{\lambda^*\left(\ell_{i-2},\hat{\rho}_{i-1}\right)}{\lambda\left(\ell_{i-2},\hat{\rho}_{i-1}\right)}\Rightarrow\rho_i\ ,$$

$$\lambda\left(\ell_{i-2},\hat{\rho}_{i-1}\right)=\sum_{j=t+1}^{\ell_{i-2}}\binom{\ell_{i-2}}{j}\hat{\rho}_{i-1}{}^{j}\left(1-\hat{\rho}_{i-1}\right)^{\ell-j}.$$

For the next iteration:

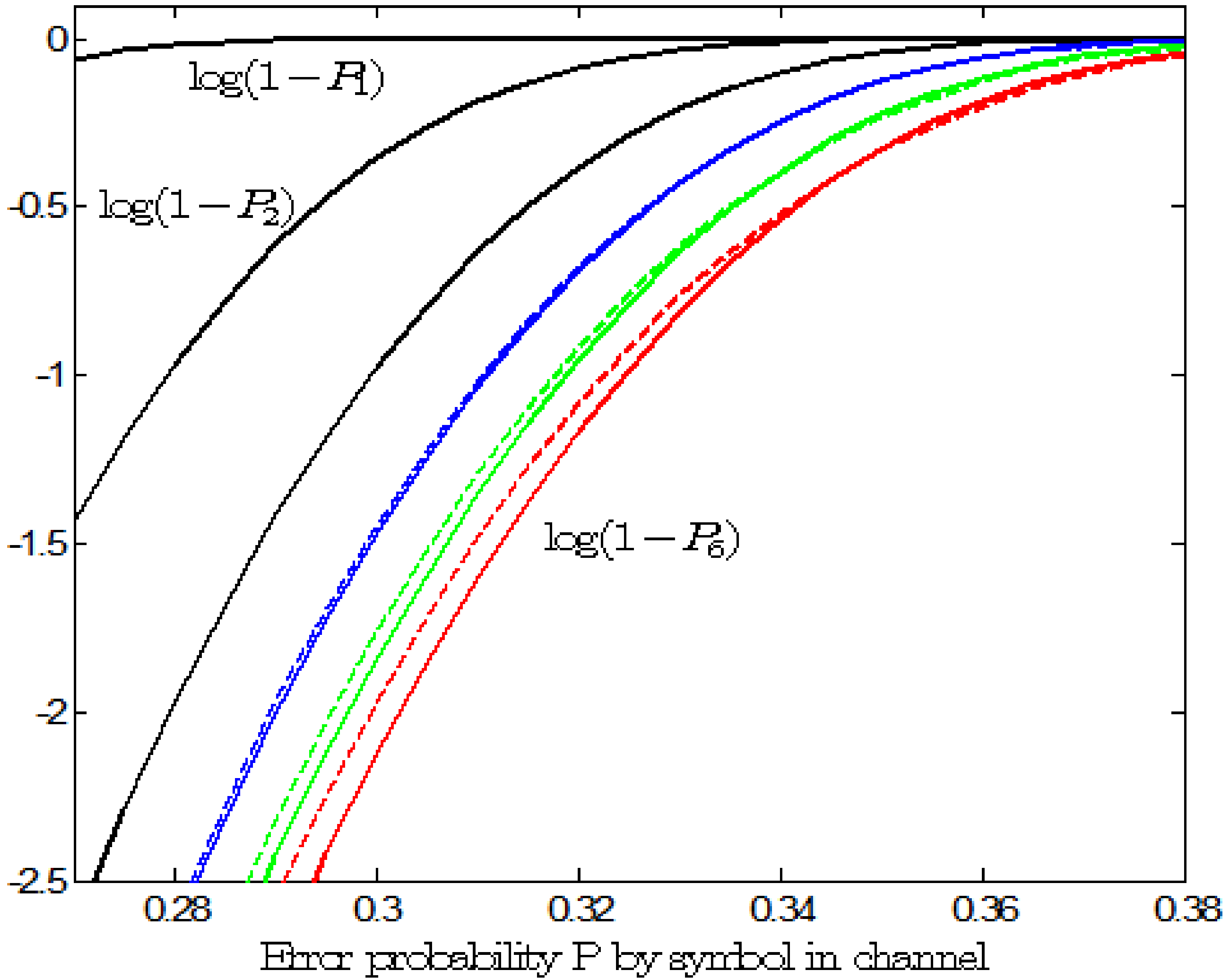
$$\Pr^{\bullet}\left(\ell_i,\ell_{i-1}\right)=\sum_{\ell_{i-2}=d+1}^{\tau(i-2)}\Pr^{\bullet}\left(\ell_{i-1},\ell_{i-2}\right)\Pr\left(\ell_i\mid\ell_{i-1},\ell_{i-2};d\le\ell_i<\ell_{i-2}\right),\ d\le\ell_i\le\tau$$

$$\Pr^{\bullet}\left(\ell_{i-2}\right)=\sum_{\ell_{i-1}=d+1}^{\tau(i-1)}\Pr^{\bullet}\left(\ell_{i-1},\ell_{i-2}\right),\ d\le\ell_{i-2}\le\tau\left(i-2\right)$$

Average density of error in "bad" block as ratio of average number of errors in that blocks to their average length.

$$\hat{\rho}_i\left(\hat{\rho}_{i-1}\right)=\frac{\sum_{\ell_{i-2}=d+1}^{\tau(i-2)}\Pr^{\bullet}\left(\ell_{i-2}\right)\rho_i\left(\ell_{i-2},\hat{\rho}_{i-1}\right)\ell_{i-2}}{\sum_{\ell_{i-2}=d+1}^{\tau(i-2)}\ell_{i-2}\Pr^{\bullet}\left(\ell_{i-2}\right)}\Rightarrow\hat{\rho}_i$$

Complexity of approximate procedure is linear on iteration number and approximately cubic on the product code size.



0
-0.5
-1
-1.5
-2
-2.5
0.28
0.3
0.32
0.34
0.36
0.38
$\log(1-P_1)$
$\log(1-P_2)$
$\log(1-P_6)$
Error probability P by symbol in channel

Conclusion

We have defined here exact and approximate procedures for calculation of probabilities of correct decoding or decoding failure for a product of Reed-Solomon codes under the strong condition that the probability of error of component decoder is negligible. The point of suggested method is definition of degradation of error model (error density ρ_i) during iterative decoding. Approximation error by iteration is small and accumulation of errors with the iteration number is a usual effect for recurrent calculation. This way for probability calculation can be expanded on other product codes.