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Are q-ary Perfect Codes reconstructed by the Vertices of Largest Level? [1]

Anastasia Vasil'eva vasilan@math.nsc.ru
Sobolev Institute of Mathematics,
Novosibirsk State University, Novosibirsk, RUSSIA

Abstract. We study the eigenfunctions on the graph of n-dimensional q-ary Hamming space. We cite the interdependence formula for local weight enumerators of an eigenfunction in two orthogonal faces. We obtain the conditions for the reconstruction of an eigenfunction into the ball by the values into the corresponding sphere.

1 Introduction

We study eigenfunctions of n-dimensional q-ary hypercube. The aim of the paper is the investigation of partial reconstruction for these functions, more precisely, of reconstruction for 1-perfect codes.

We apply an explicit formula for local distributions in two orthogonal faces [8]. The local distributions was considered in [2, 5–7] for 1-error correcting perfect codes and perfect colorings in binary case ($q = 2$). In case $q > 2$ they are investigated in [1] for 1-error-correcting codes. In [3] more general case of direct product of graphs is studied. The reconstruction problems in binary case were studied, for example, in [4, 7].

The paper is organized as follows. In Section 2 we give some necessary notations and facts, in particular, we state the formula for local weight enumerators of eigenfunctions in a pair of orthogonal faces. Furthermore, here we state the main problem. In Section 3 we carry out necessary calculations for local distributions. In Section 4 we obtain the main Theorem 2 on reconstruction for certain wide classes of eigenfunctions.

2 Eigenfunctions and local distributions

Consider the set $\mathbf{F}_q = \{0, 1, \ldots, q-1\}$ as the group by modulo q and the hypercube $\mathbf{F}_q^n$ as the abelian group $\mathbf{F}_q \times \ldots \times \mathbf{F}_q$. We investigate functions on the graph $\mathbf{F}_q^n$ of n-dimensional q-ary hypercube, in this graph two vertices are adjacent iff the Hamming distance between them equals 1.

Here and elsewhere I denotes a subset of $\{1, \ldots, n\}$ and $\overline{I} = \{1, \ldots, n\} \backslash I$. Usually we denote $k = |I|$. Take a vertex $\alpha \in \mathbf{F}_q^n$. Denote by $s(\alpha)$ the support

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of a vertex α, i.e. the set of nonzero positions of α; the cardinality of the support is equal to the Hamming weight of α. Write $W_i(\alpha)$ for the set of all vertices β that differ from α in i positions, i.e. the Hamming distance $\rho(\alpha,\beta)$ between vertices α and β is equal to i. Denote $B_i(\alpha) = W_0(\alpha)\bigcup\ldots\bigcup W_i(\alpha)$. By definition, put $\Gamma_I(\alpha) = \{\beta \in \mathbf{F}_q^n \ : \ \beta_i = \alpha_i \ \forall\, i \notin I\}$, then $\Gamma_I(\alpha)$ is said to be a k-dimensional face, it has the structure of $\mathbf{F}_q^k$. Write simply W_i and Γ_I instead of $W_i(\alpha)$ and $\Gamma_I(\alpha)$ in case α is all-zero vertex. We say that two faces $\Gamma_I(\alpha)$ and $\Gamma_J(\beta)$ are orthogonal if $J = \overline{I}$. It is easy to see that orthogonal faces have exactly one common vertex. The values of Krawtchouk polynomials $P_m^{(q)}(t;N)$ are described as coefficients of the polynomial $(x-y)^t(x+(q-1)y)^{N-t}$ and

$$P_m^{(q)}(t;N) = \sum_{j=0}^{m}(-1)^j(q-1)^{m-j}\left(\begin{array}{c} t \\ j \end{array}\right)\left(\begin{array}{c} N-t \\ m-j \end{array}\right).$$

We consider the space of all complex functions on the q-ary n-dimensional hypercube: $\left\{f : \mathbf{F}_q^n \longrightarrow C\right\}$. Any function f can be considered as vector of all values of f.

Let $D = D^{q,n}$ be an adjacency matrix of $\mathbf{F}_q^n$, i.e. $(0,1)$-matrix of order q^n where $D_{\alpha,\beta} = 1$ iff $\rho(\alpha,\beta) = 1$. It is known that the eigenvalues λ of $D^{q,n}$ are equal to $(q-1)n - qh,\ h = 0,1,\ldots,n$. For an eigenvalue λ denote by $h = h(\lambda)$ the number of λ: $h = h(\lambda) = \frac{(q-1)n-\lambda}{q}$. The corresponding eigenfunctions (we call them λ-functions) satisfy the equations

$$\sum_{\beta\in W_1(\alpha)} f(\beta) = \lambda f(\alpha),\ \alpha \in \mathbf{F}_q^n, \tag{1}$$

or in the matrix form (f is a vector of values of the function): $Df = \lambda f$.

A 1-perfect code is the subset $C \subseteq \mathbf{F}_q^n$ such that any vertex of $\mathbf{F}_q^n$ is at distance no more than 1 from exactly one vertex of the code. Therefore the function $f_C = \chi_C - \frac{1}{(q-1)n+1}$ is eigenfunction with eigenvalue $\lambda = -1$ The number of λ is equal to $h = h(-1) = \frac{(q-1)n+1}{q}$ and so $n - h = \frac{n-1}{q} \leq h$. The corresponding level of $\mathbf{F}_q^n$ is the largest.

Our aim is to study whether the intersection of any 1-perfect code with the ball can be reconstructed by the intersection with the corresponding sphere. We consider the generalized case where we take an arbitrary λ such that $n-h(\lambda)) \leq h(\lambda)$. Earlier the methods were developed in case $h \leq min\{h, n-h\}$.

By definition, for an arbitrary function f put

$$v_j^{I,f}(\alpha) = \sum_{\beta\in\Gamma_I(\alpha)\bigcap W_j(\alpha)} f(\beta),$$

the vector $v^{I,f}(\alpha) = (v_0^{I,f}(\alpha),\ldots,v_{|I|}^{I,f}(\alpha))$ is called the local distribution of the function f in the face $\Gamma_I(\alpha)$ with respect to the vertex α, or shortly (I,α)-local distribution of f. We say that the polynomial

$$g_f^{I,\alpha}(x,y)=\sum_{j=0}^{k} v_j^{I,f}(\alpha)y^j x^{k-j}=\sum_{\beta\in\Gamma_I(\alpha)} f(\beta)y^{|s(\beta)|}x^{|I|-|s(\beta)|}$$

is a (I,α)-local weight enumerator of f. For simplicity of notations we omit α if $\alpha=0=(0,\dots,0)$ and f if obvious. There is a tight interdependence between local weight enumerators of an arbitrary λ-function in two orthogonal faces:

Theorem 1. *[8] Let λ be an eigenvalue of $\mathbf{F}_q^n$, f be a λ-function, $h=\frac{(q-1)n-\lambda}{q}$ and $\alpha\in\mathbf{F}_q^n$. Put $x'=x+(q-2)y$, $y'=-y$. Then*

$$(x+(q-1)y)^{h-|\overline{I}|}g_f^{\overline{I},\alpha}(x,y)=(x'+(q-1)y')^{h-|I|}g_f^{I,\alpha}(x',y').$$

Rewrite the formula:

$$g^{\overline{I},\alpha}(x,y)=(x-y)^{h-|I|}(x+(q-1)y)^{|\overline{I}|-h}g^{I,\alpha}(x+(q-2)y,-y) \qquad (2)$$

3 Calculation of local distributions

We intend to provide an explicit formula for $(\overline{I},\alpha)$-local distribution of an arbitrary λ-function in terms of (I,α)-local distribution of this function using Theorem 1 and formula (2). The relation between the dimension $k=|I|$ of the face Γ_I and the value $h=h(\lambda)$ can be as follows:

I) $|I|\le min\{h,n-h\}$

II) $h<|I|\le|I|\le n-h$

III) $n-h<|I|\le|I|\le h$

IV) $|I|>max\{h,n-h\}$

Now we consider only the cases I) and III).

First represent the polynomial $g_f^{I,\alpha}(x+(q-2)y,-y)$ in variables x,y:

Lemma 1. *For an arbitrary λ-function f*

$$g_f^{I,\alpha}(x+(q-2)y,-y)=\sum_{l=0}^{k} y^l x^{k-l}\sum_{i=0}^{l}(-1)^i v_i^{I,f}(\alpha)(q-2)^{l-i}\left(\begin{array}{c} k-i \\ l-i \end{array}\right). \qquad (3)$$

The following lemma was obtained earlier from formulae (2) and (3):

Lemma 2. *If $|I|=k\le min\{h,n-h\}$ then for any λ-function f*

$$v_j^{\overline{I},f}(\alpha)=\sum_{i=0}^{j} r_{ij}^k v_i^{I,f}(\alpha), \qquad (4)$$

where

$$r_{ij}^k = (-1)^i \sum_{l=0}^{j-i} P_{j-i-l}^{(q)}(h-k; n-2k)(q-2)^l \left(\begin{array}{c} k-i \\ l \end{array} \right) \tag{5}$$

Pass to the third case. Let $n-h < k \le h$. Rewrite the formula (2):

$$(x+(q-1)y)^{h+k-n} g^{\overline{I},\alpha}(x,y) = (x-y)^{h-k} g^{I,\alpha}(x+(q-2)y, -y) \tag{6}$$

Denote by U the square matrix of order $s+1$ with elements for any $i,j = 0,\dots,s$

$$u_{ji} = (-1)^i (q-1)^{j-i} \left(\begin{array}{c} h+k-n \\ s-i \end{array} \right).$$

This matrix is lower triangular with nonzero diagonal elements. So there exists the inverse matrix $U^{-1} = (u'_{si})_{s,i=0\dots,j}$ that is also lower triangular.

Lemma 3. *Let $k = |I|$ and $h < k \le n-h$. Then for any $j = 0,\dots,h-k$*

$$v_j^{\overline{I},f}(\alpha) = \sum_{i=0}^{j} r_{ij}^k v_i^{I,f}(\alpha), \tag{7}$$

where u'_{js} are elements of the matrix U^{-1} and

$$r_{ij}^k = (-1)^i \sum_{s=0}^{j} u'_{js} P_{j-i}^{(q-1)}(h-k, h-i). \tag{8}$$

Proof. First note that

$$(x+(q-1)y)^{h+k-n} g^{\overline{I},\alpha}(x,y) = \sum_{s=0}^{h} y^s x^{h-s} \sum_{i=0}^{s} (-1)^i v_i^{\overline{I},f}(\alpha)(q-1)^{s-i} \left(\begin{array}{c} h+k-n \\ s-i \end{array} \right).$$

Using lemma 1 get by direct calculations

$$(x-y)^{h-k} g_f^{I,\alpha}(x+(q-2)y, -y) = \sum_{s=0}^{h} y^s x^{h-s} \sum_{i=0}^{s} (-1)^i v_i^{I,f}(\alpha) P_{s-i}^{(q-1)}(h-k, h-i).$$

For any $s = 0,\dots,h$ equate the coefficients in $y^s x^{h-s}$ for polynomials of (6):

$$\sum_{i=0}^{s} (-1)^i v_i^{\overline{I},f}(\alpha)(q-1)^{s-i} \left(\begin{array}{c} h+k-n \\ s-i \end{array} \right) = \sum_{i=0}^{s} (-1)^i v_i^{I,f}(\alpha) P_{s-i}^{(q-1)}(h-k, h-i). \tag{9}$$

In particular, this is true for any $j = 0,\dots,s$, where $s \le h$. Form the linear system with these $s+1$ equations in variables $v_i^{\overline{I},f}(\alpha)$, $i = 0,\dots,s,$. It means that our system is resolvable and the unique solution is presented in Lemma. □

Note that the expression of coefficients r_{ij}^k depends on k: it is defined as in (5) for $k \le min\{n-h, h\}$ and as in (8) for $n-h < k \le h$.

4 Reconstruction

We deal with the following question. We know the values $f(\alpha)$ for all $\alpha \in W_d$. Is it possible to determine uniquely the values $f(\alpha)$ for all $\alpha \in B_d$?

Our main goal is the case of perfect codes, i.e. $\lambda = -1,\ h = \frac{(q-1)n+1}{q}$ and $d \leq h$. Note that here $n - h < h$. Thus, we try to reconstruct λ-functions under condition $n - h < h$.

The following theorem allows us to reconstruct any λ-function into the ball by its values into the corresponding sphere under some conditions.

Theorem 2. *Let λ be an eigenvalue of $\mathbf{F}_q^n,\ n-h < h,\ d \leq h$ and $\varphi : W_d \longrightarrow C$ be a function. Suppose that f is a λ-function such that for any $\alpha \in W_d$ it holds $f(\alpha) = \varphi(\alpha)$. Then for any $\alpha \in B_d$ the value $f(\alpha)$ is uniquely determined if for all $k = 0, \ldots, d$ and $l = 0, \ldots, k$*

$$\sum_{i=0}^{k} r_{i,d-k}^{k} P_i^{(q-1)}(l,k) \neq 0, \tag{10}$$

where r_{ij}^k is defined as in (5) in case $k \leq n-h$ and in (8) in case $k > n-h$.

Proof. The proof is done by induction upon Hamming weight of vertices. The base of induction is given by the following well-known fact:

$$\sum_{\alpha \in W_d} f(\alpha) = P_d^{(q)}(h; n) f(\mathbf{0}).$$

Suppose that (10) holds and the values of f at all vertices of weight no more than $k-1$ are uniquely determined. Let I be a k-subset of $\{1, \ldots, n\}$ and a vertex α be with the support $s(\alpha) = I$. It is obvious that the support of an arbitrary vertex from the face $\Gamma_I(\alpha)$ is included in I. Consider the component $v_i^{I,f}(\alpha),\ i \leq k$, of the (I, α)-local distribution of the λ-function f. This component can be decomposed into two sums $\delta_i^{I,f}(\alpha)$ and $\sigma_i^{I,f}(\alpha)$, where the summation is done over all vertices with the support I and over all vertices with the support less than I. By assumption, the values of $\sigma_i^{I,f}(\alpha)$ are already known. Therefore we can write the linear system for the values of $\delta_i^{I,f}(\alpha)$, the form of these equations follows from Lemmas 2 3. They compose the linear system with the matrix

$$M^{d,k} = \sum_{i=0}^{k} r_{i,d-k}^{k} D_i^{q-1,k} \tag{11}$$

The system has the unique solution iff its matrix $M^{d,k}$ has the full rank. (Note that a solution exists by virtue of the hypothesis of Theorem.) Then represent incidence matrices in terms of primitive idempotents $J_l^{q-1,k}$ of Hamming accosiation scheme. So the matrix

$$M^{d,k} = \sum_{i=0}^{k} r^k_{i,d-k} \sum_{l=0}^{k} P_i^{(q-1)}(l,k) J_l^{q-1,k} = \sum_{l=0}^{k} J_l^{q-1,k} \sum_{i=0}^{k} r^k_{i,d-k} P_i^{(q-1)}(l,k).$$

has the full rank iff all coefficients in primitive idempotents are nonzero. □

References

[1] Soohak Choi, Jong Y. Hyun, Hyun K. Kim: Local duality theorem for q-ary 1-perfect codes. Designs, Codes and Cryptography, vol. 70, issue 3, pp. 305-311. (2014)

[2] Jong Y. Hyun: Local duality for equitable partitions of a Hamming space. J. Comb. Theor., Ser. A, vol.119, no. 2, pp. 476-482 (2012)

[3] Krotov D.S., On weight distributions of perfect colorings and completely regular codes. Designs, Codes and Cryptography, vol. 61, issue 3, pp 315-329 (2011)

[4] Heden Olof: On the reconstruction of perfect codes. Discrete Mathematics, vol. 256, no. 1-2, pp. 479-485 (2002)

[5] Vasil'eva A. Yu.: Local spectra of perfect binary codes. Discrete Applied Mathematics, vol. 135, no. 1-3, pp. 301-307 (2004)

[6] Vasil'eva A.Yu.: Local and Interweight Spectra of Completely Regular Codes and of Perfect Colourings. Probl. Inform. Transm., vol. 45, no. 2, pp. 151-157 (2009).

[7] Vasil'eva A.Yu.: Local Distribution and Reconstruction of Hypercube Eigenfunctions. Probl. Inform. Transm., vol. 49, no. 1, pp. 32-39 (2013).

[8] Vasil'eva A.: Local distributions of q-ary eigenfunctions and of q-ary perfect colorings. Proceedings of Seventh International Workshop on Optimal Codes and Related Topics OC2013, Sofia: Inst. of Math. and Informatics, pp. 181-186 (2013)