Abstract. In the projective plane $PG(2,q)$, upper bounds on the smallest size $t_2(2,q)$ of a complete arc are considered. The results of computer search for a huge region of $q$, done in the recent works of the authors and in this work, are investigated. New upper bounds valid in this region are proposed. Our investigations and results allow us to conjecture that these bounds hold for all $q$.

1 Introduction

Let $PG(2,q)$ be the projective plane over the Galois field $F_q$. An $n$-arc is a set of $n$ points no three of which are collinear. An $n$-arc is called complete if it is not contained in an $(n+1)$-arc of $PG(2,q)$ [1–7].

One of the most important open problems in the study of projective planes is the determination of the smallest size $t_2(2,q)$ of a complete arc in $PG(2,q)$. This work is devoted to upper bounds on $t_2(2,q)$. 
Let \( t(\mathcal{P}_q) \) be the size of the smallest complete arc in any (not necessarily Galois) projective plane \( \mathcal{P}_q \) of order \( q \). In [7], for sufficiently large \( q \), the following result is proven by probabilistic methods:
\[
t(\mathcal{P}_q) \leq D\sqrt{q} \ln^C q, \quad C \leq 300,
\]
where \( C \) and \( D \) are constants independent of \( q \). The authors of [7] conjecture that the constant can be reduced to \( C = 10 \). We denote
\[
T = \{ q : 173 \leq q \leq 49727, \ q \text{ power prime} \} \cup \{ q : 173 \leq q \leq 125003, \ q \text{ prime} \} \\
\cup \{ 59 \text{ sporadic prime } q \text{'s in the interval } [125101 \ldots 360007] \}; \\
T^\# = \{ q : 125017 \leq q \leq 150001, \ q \text{ prime} \} \cup \{ 290011, 370003, 400007 \}; \\
L = \{ q \leq 67993, \ q \text{ prime} \} \cup \{ 43 \text{ sporadic prime } q \text{'s in } [69997 \ldots 190027] \}; \\
L^\# = \{ 152501, 157513, 162517, 167521, 172507, 195023, 200003, 205019, 210011, 215051, 220009, 225023, 230003, 235003, 240007 \}; \\
R = \{ q \leq 46337, \ q \text{ prime} \}.
\]

Let \( t_2(2, q) \) be the smallest known size of a complete arc in \( PG(2, q) \). For \( q \in T \), the values of \( t_2(2, q) \) (up to November 2013) are collected in [2]. In [4,5] the algorithm FOP (Fixed Order of Points) used to construct small complete arcs in \( PG(2, q) \) is described. Let \( t^L_2(2, q) \) be the size of a complete arc in \( PG(2, q) \) obtained by Algorithm FOP with Lexicographical order of points. For \( q \in L \), the sizes \( t^L_2(2, q) \) are collected in [4]. Let \( t^R_2(2, q) \) be the size of a random complete arc in \( PG(2, q) \). For \( q \in R \), the sizes \( t^R_2(2, q) \) are given in [6].

In this work, by computer search\(^1\) using randomized greedy algorithms similar to those in [2,3], the values of \( t_2(2, q) \) are obtained for \( q \in T^\# \). Also, using Algorithm FOP we obtained the arc sizes \( t^L_2(2, q) \) for \( q \in L^\# \).

Investigating the results of [1–5] for \( q \in T \) and \( q \in L \) and the results of this work for \( q \in T^\# \) and \( q \in L^\# \), we obtained Theorems 2 and 3.

**Conjecture 1.** Bounds (1) and (2) of Theorems 2 and 3 hold for all \( q \geq 109 \).

### 2 \ Bounds based on greedy algorithms’ results

**Theorem 2.** In \( PG(2, q) \), for the smallest size \( t_2(2, q) \) of a complete arc, the following upper bounds hold:
\[
t_2(2, q) < \min\{ \sqrt{q} \ln^{0.7295} q, \ \sqrt{q} \ln^{f(q)} q, \ 0.6\sqrt{q} \ln^{\varphi(q)} q \}, \tag{1}
\]
\[
109 \leq q \leq 169 \quad \text{and } q \in T \cup T^\#,
\]
where \( f(q) = \frac{0.27}{\ln q} + 0.7 \), \( \varphi(q) = \frac{1.5}{\ln q} + 0.802 \).

\(^1\)The calculations are done using computational resources of Multipurpose Computing Complex of National Research Centre “Kurchatov Institute” (http://computing.kiae.ru/).
Arcs satisfying these bounds can be obtained by the randomized greedy algorithms described in [2, 3].

The statement of Theorem 2 is illustrated by Figures 1–4 where the values \( \varphi^*(q), f^*(q), \varphi(q), \) and \( \overline{f}(q) \) are defined by relations \( \varphi^*(q) = \varphi(q) - 0.5, f^*(q) = f(q) - 0.5, \overline{t}_2(2, q) = 0.6 \sqrt{q} \ln \varphi(q) q, \overline{t}_2(2, q) = \sqrt{q} \ln \overline{f}(q) q. \)
Figure 2: Upper bounds on $t_2(2, q)$ vs greedy algorithms’ results (II)
ln$^{0.2295}$ q (the top dashed curve); ln$f^*(q)$ q (the 2-nd dashed-dotted curve);
0.6 ln$^{0.27}$ q (the 3-rd dashed-dotted curve); values $\frac{t_2(2, q)}{\sqrt{q \ln q}}$ from complete arcs
obtained by the greedy algorithms, $q \in T \cup T^\#$ (the bottom solid curve).

3 A bound based on results of Algorithm FOP

Algorithm FOP. Consider the projective plane $PG(2, q)$ and fix a particular
order on its points. The algorithm builds a complete arc iteratively. Let $K^{(i-1)}$
be the arc obtained on the $(i-1)$-th step. On the next step, the first point in
the fixed order not lying on the bisecants of $K^{(i-1)}$ is added to $K^{(i-1)}$. Suppose
that the points of $PG(2, q)$ are ordered as $A_1, A_2, \ldots, A_{q^2+q+1}$. Consider the
empty set as root of the search and let $K^{(j)}$ be the partial solution obtained in
the $j$-th step, as extension of the root. We put $K^{(0)} = \emptyset$,
$K^{(1)} = \{A_1\}$, $K^{(2)} = \{A_1, A_2\}$, $m(1) = 2$, $K^{(j+1)} = K^{(j)} \cup \{A_{m(j)}\}$, $m(j) = \min\{i \in [m(j-1) + 1, q^2 + q + 1] | \not\exists P, Q \in K^{(j)} : A_i, P, Q \text{ are collinear}\}$,
that is $m(j)$ is the minimum subscript $i$ such that the corresponding point $A_i$
is not saturated by $K^{(j)}$. The process ends when a complete arc is obtained.
Lexicographical order of points. Let $q$ be prime and let the elements of the field $\mathbb{F}_q = \{0, 1, \ldots, q - 1\}$ be treated as integers modulo $q$. Let the points $A_i$ of $PG(2, q)$ be represented in homogenous coordinates so that $A_i = (x_0^{(i)}, x_1^{(i)}, x_2^{(i)})$, $x_j^{(i)} \in \mathbb{F}_q$, where the leftmost non-zero element is 1. For $A_i$, we put $i = x_0^{(i)} q^2 + x_1^{(i)} q + x_2^{(i)}$. So, the homogenous coordinates of a point $A_i$ are treated as its number $i$ written in the $q$-ary scale of notation.

**Theorem 3.** In $PG(2, q)$, for the smallest size $t_2(2, q)$ of a complete arc, the following upper bound holds:

$$t_2(2, q) < 1.83 \sqrt{q \ln q}, \quad q \in L \cup L^#.$$  \hfill (2)

Arcs satisfying these bounds can be obtained by Algorithm FOP with fixed Lexicographical order of points.

The statement of Theorem 3 is illustrated by Figure 5.
Figure 4: The function $f(q)$ vs $\overline{f}(q)$ obtained by greedy algorithms. $y = 0.7295$ (the top dashed line); $f(q)$ (the 2-nd dashed-dotted curve); values $\overline{f}(q)$ from complete arcs obtained by greedy algorithms (the bottom solid curve).

**Lemma 4.** It holds that
\[
\min\{\sqrt{q}\ln^{0.7295} q, \sqrt{q}\ln^{f(q)} q, 0.6\sqrt{q}\ln^{\phi(q)} q\} < 1.83\sqrt{q}\ln q.
\]

**Remark 5.** In Coding Theory, greedy codes (or lexicographical codes, or lexicodes) are considered, see [8,9] and the references therein. In [5, Remark 2.1], it is noted that formally Algorithm FOP is an algorithm creating a parity check matrix of a lexicode with codimension 3 and minimum distance 4.

### 4 Random complete arcs in $PG(2, q)$

In [6] small complete arcs are obtained using an algorithm which randomly selects at each step the point to add. The corresponding values $t_2^R(2, q)/\sqrt{q}\ln q$, $q \in R$, are shown in Figure 6. One can see that Figures 5 and 6 have a very similar structure. This is expected, as Lexicographical order of points is a random order in the geometrical sense.
Figure 5: Bound $1.83\sqrt{q \ln q}$ vs Algorithm's FOP results. $y = 1.83$ (the top solid line); $y = 1.803$ (the 2-nd dashed line); $t_L^*(2,q)/\sqrt{q \ln q}$ obtained by Algorithm FOP with Lexicographical order of points, $q \in L \cup L^\#$ (the solid curve).

References


Figure 6: **Bound $1.83\sqrt{\frac{q}{\ln q}}$ vs random complete arcs.** $y = 1.83$ (the top solid line); $y = 1.803$ (the 2-nd dashed line); values $t_2^R(2,q)/\sqrt{q\ln q}$ from random complete arcs, $q \in R$ (the solid curve).


