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On the Erasure-Correcting Capabilities of Irregular LDPC Codes [1]

Pavel Rybin prybin@iitp.ru
Institute for Information Transmission Problems, Russian Academy of Science, Moscow 101447, Russia

Abstract. This paper deals with the irregular binary low-density parity-check (LDPC) codes with constituent single parity check (SPC) codes and erasure-correcting iterative low-complex decoding algorithm. The lower bound on the erasure fraction, guaranteed corrected by the considered iterative algorithm, was obtained for irregular LDPC code for the first time in this paper. This lower bound was obtained as a result of analysis of Tanner graph representation of irregular LDPC code. The number of decoding iterations, required to correct the erasures, is a logarithmic function of the code length. The numerical results, obtained at the end of the paper for proposed lower bound achieved similar results for previously known best lower-bounds for regular LDPC codes and were represented for the first time for irregular LDPC codes.

1 Introduction

The erasure correcting capabilities of Gallager's low-density parity-check (LDPC) [1] for binary erasure channel (BEC) were studied in [2], where it was shown that such Gallager's LDPC code exists that capable of correcting a linear portion of erasures, with decoding complexity $O(n \log n)$, where n – LDPC code length. Then the result similar to the result from [2] for the first time was obtained for LDPC code with constituent Hamming code in [3] by generalization of the methods developed in [4]. Then using generalized methods from [3] the new lower-bound for Gallager's LDPC code was significantly improved in [5].

This work was inspired partly by [5, 6] and partly by [7]. In this paper we consider the irregular LDPC codes and erasure-correcting iterative low-complex decoding algorithm similar to the algorithm from [2, 5]. We obtained new lower-bound on fraction of guaranteed corrected erasures generalizing and combining the methods, developed in [5, 7]. Numerical computation for various choices of LDPC code shows that proposed lower-bound achieves similar results for previously known best lower bounds for regular LDPC codes [5]. The numerical results for irregular LDPC codes are represented for the first time.

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2 LDPC code construction

It is convenient to specify LDPC codes using their Tanner graph representation [8]. The Tanner graph is a bipartite graph, where the nodes on the left side are associated with the code-word bits (variable nodes) and the nodes on the right are associated with the parity-check equations (check nodes). The irregular LDPC code ensemble considered in this paper is based on the following ensemble of irregular bipartite graphs. It is characterized by two probability vectors

$$\tilde{\lambda} = \left(\tilde{\lambda}_2, \ldots, \tilde{\lambda}_c\right), \tilde{\rho} = \left(\tilde{\rho}_1, \ldots, \tilde{\rho}_d\right),$$

where $\tilde{\lambda}_l$ is the fraction of variable nodes with degree l, and $\tilde{\rho}_l$ is the fraction of check nodes with degree l. For convenience we also define the polynomials

$$\tilde{\lambda}(x) = \sum_{l=2}^{c} \tilde{\lambda}_l x^{l-1}, \tilde{\rho}(x) = \sum_{l=2}^{d} \tilde{\rho}_l x^{l-1}.$$

Let E denotes the total number of edges, n denotes the number of left nodes and m denotes the number of right nodes. Then

$$n = \frac{E}{\sum_{l=2}^{c} \tilde{\lambda}_l l} = \frac{E}{1 + \tilde{\lambda}'(1)}, m = \frac{E}{\sum_{l=2}^{d} \tilde{\rho}_l l} = \frac{E}{1 + \tilde{\rho}'(1)},$$

where $\tilde{\lambda}'(1)$ and $\tilde{\rho}'(1)$ are derivatives of functions $\tilde{\lambda}(x)$ and $\tilde{\rho}(x)$ of variable x calculated in the point $x = 1$.

For each variable node with degree i we assign i variable sockets. Similarly, for each check node with degree i we assign i check sockets. The total number of variable sockets and the total number of check sockets are both equal to the total number of edges E. The ensemble of bipartite graphs is obtained by choosing a permutation π with uniform probability from the space of all permutations of size E. For each $1 \leq i \leq E$, we connect the variable node associated with the ith variable socket to the check node associated with the π_ith check socket. Note that in this way, multiple edges may link a pair of nodes. The mapping from the bipartite graph space to the parity-check matrix $\mathbf{H}$ space is such that an element $\mathbf{H}_{i,j}$ in the matrix, corresponding to the ith check node and jth variable node, is set to "1" if there is an odd number of edges between the two nodes, and to "0" otherwise.

The rate R' of each code in the ensemble satisfies $R' \geq R$, where

$$R = 1 - \frac{m}{n} = 1 - \frac{\sum_{l=2}^{c} \tilde{\lambda}_l l}{\sum_{l=2}^{d} \tilde{\rho}_l l} = 1 - \frac{1 + \tilde{\lambda}(1)}{1 + \tilde{\rho}'(1)}$$

is the planned rate of the code (the inequality is due to a possible degeneracy in the m parity-check equations).

A special case of the irregular code ensemble that was described above is obtained when all variable nodes have a constant degree c and all check nodes have a constant degree d. In this case, the ensemble is regular, $nc = md$ and $R = 1 - c/d$.

3 Decoding algorithm

Let us now consider the decoding algorithm $\mathcal{A}$ of LDPC code with constituent single parity check (SPC) code. It is the same as the algorithm from [2] and [5]. The main idea is to find at least one SPC code with one erasure on each decoding iteration and correct it. In this case the number of erasures in the tentative sequence decreases with iteration. And due to the finite number of erasures in received sequence they will be corrected in finite number of iterations.

It is important to note that unlike the correction of errors the correction of erasures doesn't add new erasures to the tentative sequence and positions of current erasures are known. Thus to correct all erasures the parity-check with correctable combination of erasures must exists on each iteration. Let e_i denotes the number of check nodes corresponding to parity-checks with correctable combination of erasures connected to the ith variable node. Then the current erased ith symbol will be corrected if $e_i > 0$. Let l_i denotes the degree of ith variable node. Than the following condition guarantees that for given LDPC code and given erasure pattern with W erasures the parity checks with correctable combination of erasures exists:

$$E_W = \sum_{j=1}^{W} e_{i_j} > \alpha \sum_{j=1}^{W} l_{i_j}, \tag{1}$$

where W is number of erasures in received sequence, i_1, i_2, $\ldots$, i_W are the positions of erased symbols and α is some small constant, $0 \le \alpha \le 1$.

Remark 1. *The α is arbitrary small constant (linear portion of parity checks with correctable combination of erasures). This constant affects the estimation of decoding complexity. If $\alpha > 0$ the complexity of decoding algorithm $\mathcal{A}$ is shown to be $\mathcal{O}(n \log n)$ similarly to the [2] and [5].*

Remark 2. *By estimation of the probability of condition (1) realization, we obtain lower bound on the fraction of guaranteed corrected errors, represented in the next section.*

4 Main result

Theorem 1. *Let exist the positive root ω_0 of the following equation:*

$$\max_{0\le\beta\le\gamma}\left\{\tau\left(\omega,\beta\right)+\theta\left(\gamma,\alpha,\beta\right)-\gamma h\left(\frac{\beta}{\gamma}\right)\right\}=0,$$

where $h\left(\frac{\beta}{\gamma}\right)=-\frac{\beta}{\gamma}\log_2\frac{\beta}{\gamma}-\left(1-\frac{\beta}{\gamma}\right)\log_2\left(1-\frac{\beta}{\gamma}\right)$ *is binary entropy function,* $\gamma=\sum_i i\tilde{\lambda}$ *is the average degree of a variable node,* $\tau\left(\omega,\beta\right)$ *is given by*

$$\tau\left(\omega,\beta\right)=\min_{\substack{x>0\\y>0}}\left\{\log_2\left(1+xy\tilde{\lambda}\left(y\right)\right)-\omega\log_2 x-\beta\log_2 y\right\}$$

and $\theta\left(\gamma,\alpha,\beta\right)$ *is given by*

$$\begin{aligned}\theta\left(\gamma,\alpha,\beta\right)=\min_{\substack{x>0\\0<y<1}}\ &\{(1-R)\log_2\left(\gamma x\left(y-1\right)\right.\\&+\ \left.(1+x)\,\tilde{\rho}\left(1+x\right)\right)-\beta\log_2 x-\alpha\beta\log_2 y\},\end{aligned}$$

where $\alpha>0$ *is arbitrary small constant (linear portion of parity checks with only one erasure).*

Then such irregular LDPC code with degree polynomials $\tilde{\lambda}\left(x\right)=\sum_{l=2}^{c}\tilde{\lambda}_l x^{l-1}$ *and* $\tilde{\rho}\left(x\right)=\sum_{l=2}^{d}\tilde{\rho}_l x^{l-1}$ *exists (with probability* $p_n:\lim\limits_{n\to\infty}p_n=1$*), which can correct any erasure pattern with weight less than* $\lfloor\omega_0 n\rfloor$ *with decoding complexity* $\mathcal{O}\left(n\log n\right)$*.*

Due to the space limitation, the proof is omitted. But the main idea is similar to the idea of the proof from [9].

5 Numerical results

In this section the numerical results, obtained using proposed lower bound, are represented for some parameters of irregular LDPC codes. In table 1 the numerical results for irregular LDPC codes with $R=1/2$ and for the given degree polynomials are represented. In table 1 you can notice that in irregular case the results doesn't exceed the results for regular case, but with growth of average degree of variable node this difference decreases. In table 2 the numerical results for the case of irregular LDPC code with average variable degree equal to 10 are represented in more details. As you can see addition the nodes with small degree leads to the decreasing of guaranteed corrected erasure fraction.

Table 1: Numerical results for irregular LDPC codes with $R = 1/2$ and given degree polynomials

$\tilde{\lambda}_5$	0	0.25	0	0	0	0
$\tilde{\lambda}_{10}$	1	0.5	0	0	0	0
$\tilde{\lambda}_{15}$	0	0.25	0	0.25	0	0
$\tilde{\lambda}_{20}$	0	0	1	0.5	0	0
$\tilde{\lambda}_{25}$	0	0	0	0.25	0	0.25
$\tilde{\lambda}_{30}$	0	0	0	0	1	0.5
$\tilde{\lambda}_{35}$	0	0	0	0	0	0.25
$\tilde{\rho}_{20}$	1	1	0	0	0	0
$\tilde{\rho}_{40}$	0	0	1	1	0	0
$\tilde{\rho}_{60}$	0	0	0	0	1	1
ω_0	6.2e-2	6.1e-2	4.6e-2	4.5e-2	3.6e-2	3.6e-2

Table 2: Numerical results for irregular LDPC codes with $R = 1/2$ and fixed $\tilde{\lambda}_{10} = 0.5$ and $\tilde{\rho}_{20} = 1$

$\tilde{\lambda}_5$	0.25	0.3333	0.375	0.4	0.4167	0.4286
$\tilde{\lambda}_{10}$	0.5	0.5	0.5	0.5	0.5	0.5
$\tilde{\lambda}_{15}$	0.25	0	0	0	0	0
$\tilde{\lambda}_{20}$	0	0.1667	0	0	0	0
$\tilde{\lambda}_{25}$	0	0	0.125	0	0	0
$\tilde{\lambda}_{30}$	0	0	0	0.1	0.5	0.5
$\tilde{\lambda}_{35}$	0	0	0	0	0.0833	0
$\tilde{\lambda}_{40}$	0	0	0	0	0	0.0714
ω_0	6.1e-2	5.9e-2	5.4e-2	4.9e-2	4.6e-2	4.2e-2

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