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The Fast Block Circulant Jacket Transform [1]

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Abstract. In this paper, we propose a note on the fast block circulant Jacket matrices of orders $N = 2p, 4p, 4^k p, np$, where k is a positive integer, for MIMO block diagonal channel. A class of block Toeplitz circulant jacket matrices, not only possess many properties of the Walsh-Hadamard Toeplitz transform, but also have the construction of block circulant, which can apply fast algorithms for decomposition easily. The matrix decomposition is of the form of the products of block identity I_2 matrices and block Hadamard H_2 matrices. Motivation of this paper, we assume a block fading channel model, where the channel is constant during a transmission block and changes independently between consecutive transmission blocks, can achieve a better performance in high SNR compared with i.i.d channel. This algorithms for realizing these transforms can be applied to the Kronecker MIMO channel.

1 Introduction

Discrete orthogonal transforms, such as Walsh-Hadamard transform (WHT), discrete ourier transform (DFT) play a key important role in digital signal and image processing application, since orthogonal transforms can often decorrelate the components of a given signal and redistribute the energy contained in the signal so that most of energy is contained in a small number of components[1-8]. For example, the Walsh-Hadamard transform(WHT) has been widespreadly using in signal processing[1-9], image processing[2], error-correcting codes[3], and orthogonal design[4], because of fast computational algorithm is very simplicity, i.e. +1,-1. The researcher have made a considerable amount of effort to develop various kind of discrete orthogonal transforms. Since the orthogonal transform with the independent parameters can carry many different characterisation of digital signals, it is interesting to investigate the possibility of generalization of WHT which is the Jacket transform. Generalized, the Jacket matrices idea is from center weighted Hadamard matrices[7,8].

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The proposed transforms with special structure are desirable. Although there exist circulant Hadamard matrices of order 1 and order 4, one conjectures that there do not exist any circulant Hadamard matrices of order at least 5. In this paper, we develop some method for construction of block Toeplitz circulant jacket transform, and it can be applied to the Kronecker MIMO channel.

Definition 1. *Let $A = (a_{jk})$ be an $n \times n$ matrix whose elements are in a field $\mathcal{F}$ (including real fields, complex fields, finite fields, etc.). Denote by $A^\dagger$ as the transpose matrix of the element-wise inverse of A, that is, $A^\dagger = (a_{jk}^{-1})$. Then A is called a Jacket matrix if $AA^\dagger = A^\dagger A = nI_n$, where I_n is the identity matrix over the field $\mathcal{F}$. Then, we can easily calculate the inverse of large matrices.*

For example, given a matrix, A, and its element-wise inverse transpose, $A^\dagger$, as

$$A = \begin{pmatrix} a & \sqrt{ac} \\ \sqrt{ac} & -c \end{pmatrix}, A^\dagger = \begin{pmatrix} \frac{1}{a} & \frac{1}{\sqrt{ac}} \\ \frac{1}{\sqrt{ac}} & -\frac{1}{c} \end{pmatrix}, \tag{1}$$

we say A is a 2×2 Jacket matrix. As a special case, when $a = c = 1$, A reduces to a 2×2 Hadamard matrix.

Definition 2. *Let A be an $n \times n$ matrix. If there exists a Jacket matrix J such that $A = JDJ^{-1}$, where D is a diagonal matrix, then we say that A is Jacket similar to the diagonal matrix D[8]. We say that A is Jacket diagonalisable[25].*

Definition 3. *Let $[C]_N = \begin{pmatrix} C_0 & C_1 \\ C_1 & C_0 \end{pmatrix}$ be 2×2 block matrix of order $N = 2p$. If $[C_0]_p$ and $[C_1]_p$ are $p \times p$ jacket matrices, then $[C]_N$ is block circulant Jacket matrix if and only if $C_0 C_1^{RT} + C_1^{RT} C_0 = [0]_N$, where RT is reciprocal transpose.*

On the while, let X_n be a discrete time random process with expectation $E(X_i) = m_i$ and covariance function $Cov(X_i, X_j) = E((X_i) - m_i)(X_j j) - m_j)$. One of the interesting progress is that m_i is constant which is not dependent on i and $K_n = [Cov(X_i, X_j); i, j = 1, \ldots, n]$ is Toeplitz matrix. This result deals with the asymptotic behavior of the eigenvalues of an $n \times n$ Hermitian Toeplitz matrix $T_n = (t_{j-k})_{j,k=1}^n$, where the complex numbers t_k with $k \in \mathbb{Z}$ are the Fourier coefficients of a bounded function called the symbol of the sequence T_n [9].This progress is called weakly stationary, which has many application in progressing theory. There is a common special case circulant matrix of Toeplitz matrices, which plays a fundamental role in developing more general results. However, there are random progress whose covariance matrix is not circulant matrices, but is block circulant matrices. With the result of the above results, we are able to analysis the behavior of the eigenvalues and other properties.

Therefore, If A is Jacket similar to the diagonal matrix D, then the main diagonal entries of D are all eigenvalues. Furthermore, since $A = JDJ^{-1}$, we have $D = J^{-1}AJ$. Note that since J is a Jacket matrix, its inverse is easy to obtain, that is, $J_{n\times n}^{-1} = \frac{J_{n\times n}^{\dagger}}{n}$. Hence we can directly calculate the eigenvalues of A by $D = \frac{J^{\dagger}AJ}{n}$. In this paper, we investigate a class of matrices that may have eigenvalue decomposition (EVD) through Jacket matrices. Then we discuss its application to precoding and decoding of distributive-MIMO channels in wireless communications.

The remainder of this paper is organised as follows. The next section states fast algorithm and applications. Section 3 drwas some conclusions.

2 The fast block circulant Jacket transform

The Hadamard transform, y, of a $2^n \times 1$ vector x is defined as

$$y = H_n x, \tag{2}$$

Straightforward calculation of (2) requires $O(2^{2n})$ arithmetic operation. There are fast methods just as there are fast methods for calculate similar transforms such as the Fourier transform. On the while, the fast BCJT is he similar fashion as fast Hadamard tansform,i.e., $[C]_{np} = [I]_n \otimes C_0 + P \otimes C_1 + P^2 \otimes C_1 + \cdots + P^{n-1} \otimes C_1$, $H_N = (H_2 \otimes I_{N/2})(I_2 \otimes H_{N/2})$.

Let C_0 and C_1 be two $p \times p$ jacket matrices. Let

$$[C]_{np} = \begin{pmatrix} C_0 & C_1 & \cdots C_1 & C_1 \\ C_1 & C_0 & \cdots C_1 & C_1 \\ \dots & \dots & \ddots \dots & \dots \\ C_1 & C_1 & \cdots C_1 & C_0 \end{pmatrix}$$

be the block Jacket circulant matrix.

Furhter $[C]_{np}$ can be rewritten as

$$[C]_{np} = [I]_n \otimes C_0 + P \otimes C_1 + P^2 \otimes C_1 + \cdots + P^{n-1} \otimes C_1.$$

As an example,

i) $n = 2$ case,

$$[C]_{2p} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes C_0 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes C_1 = \begin{bmatrix} C_0 & C_1 \\ C_1 & C_0 \end{bmatrix} \tag{3}$$

Let

$$C_0 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; C_1 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix},$$

$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be the permutation matrix

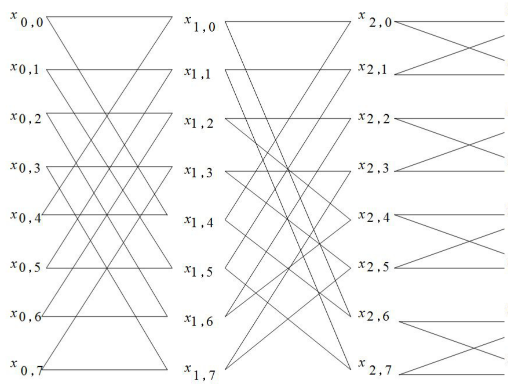

Figure 1: Fast BCJT signal flow graph for the forward, $N = 8$

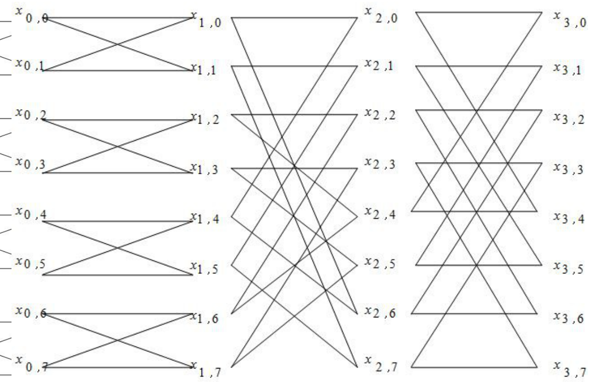

Figure 2: Fast BCJT signal flow graph for the backward, $N = 8$

ii) $n = 2, p = 2$ case,

$$\begin{aligned}
[C]_4 &= [I]_2 \otimes C_0 + P \otimes C_1 \\
&= \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{bmatrix} \begin{bmatrix} H_2 & 0 \\ 0 & H_2 \end{bmatrix} \\
&= \left[H_2 \otimes \begin{bmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{bmatrix} \right] [I_2 \otimes H_2] \qquad (4)
\end{aligned}$$

iii) $n = 4, p = 2$ case,

$[C]_8 = [I]_4 \otimes C_0 + P \otimes C_0 + P^2 \otimes C_0 + P^3 \otimes C_1$, $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ be the permutation matrix.

$$[C]_8 = ([H]_2 \otimes [I]_4) \begin{bmatrix} I_2 \otimes I_2 & I_2 \otimes I_2 \\ \hat{P}_2 \otimes I_2 & \bar{P}_2 \otimes I_2 \end{bmatrix} (I_4 \otimes H_2), where \quad \hat{P}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \bar{P}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \qquad (5)$$

Therefore,

$$
\begin{aligned}
[C]_8 &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}
\end{aligned} \tag{6}
$$

Also, for the backward transform, we have

$$
\begin{aligned}
[C]_8^{-1} &= \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \\
&= \frac{1}{8}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}
\end{aligned} \tag{7}
$$

Note that $[C]_8[C]_8^{-1} = 1$.

Then, we can get a general formula,

$$
\begin{aligned}
[C]_N &= [I]_4 \otimes C_0 + P \otimes C_0 + \ldots + P^{N/2-2} \otimes C_0 + P^{N/2-1} \otimes C_1, where \quad N = np \\
&= ([H]_2 \otimes [I]_{N/2}) \begin{bmatrix} I_2 \otimes I_{N/2} & I_2 \otimes I_{N/2} \\ \hat{P}_2 \otimes I_{N/2} & \bar{P}_2 \otimes I_{N/2} \end{bmatrix} \cdots (I_{N/2} \otimes H_2)
\end{aligned} \tag{8}
$$

Now we compute the complexity of this decomposed factor of the proposed transform. Firstly, for the block matrix C_0, there needs a multiplications and b additions. We can compare the computation complexity in the fast algorithm and the direct computation. For the order$=4p$, since in $C_0(X_0+X_2)+C_1(X_1+X_3)$, there needs $2b+3p$ additions and $2a$ multiplications. Then for the order$=4^2p$, there needs $2(2b+3p)+3p$ additions and $2 \cdot 2a$ multiplications. Note that $2(2b+3p)+3p = 2^2b+3p(2+1) = 2^2b+3p(2^2-1)$. Hence for C_{4^kp} needs $2^kb+3p(2^k-1)$ additions and 2^ka multiplications. However for C_{np}, there needs $2b+(p-1)n$ additions and $2a$ multiplications. The Table 1 show that the complexity of the proposed is better than conventional one.

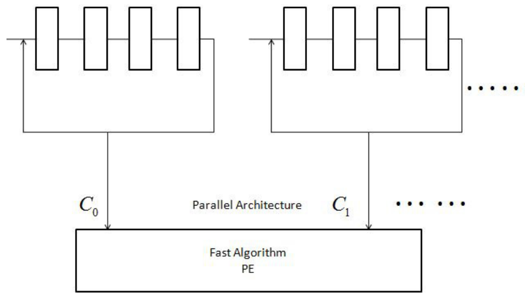

Figure 3: Architecture of the fast encoding algorithm. Matrix factorization and combinations.

In the order of $4p$,we can see

$$[C]_{4p} = \begin{pmatrix} C_0 & C_1 & C_0 & -C_1 \\ -C_1 & C_0 & C_1 & C_0 \\ C_0 & -C_1 & C_0 & C_1 \\ C_1 & C_0 & -C_1 & C_0 \end{pmatrix}.$$

In this block circulant Jakcet matrix, we have

$$[C]_{4p}X = \begin{pmatrix} C_0 & C_1 & C_0 & -C_1 \\ -C_1 & C_0 & C_1 & C_0 \\ C_0 & -C_1 & C_0 & C_1 \\ C_1 & C_0 & -C_1 & C_0 \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} C_0(X_0+X_2)+C_1(X_1-X_3) \\ C_0(X_1+X_3)+C_1(-X_0+X_2) \\ C_0(-X_0+X_2)+C_1(-X_1+X_3) \\ C_0(X_1+X_3)+C_1(X_0-X_2) \end{pmatrix}.$$

It is very clear that we have $2b+3p$ additions and $2a$ multiplications in $C_0(X_0+X_2)+C_1(X_1+X_3)$, and same in the others. The proposed hardware implementation is shown in Fig.3. The shift register provides the block circulant matrix units, and fast algorithm products the whole matrix by using proper construction.

Table 1: Comparison of complexity with conventional and proposed $4p, 4^k p, np$

	Direct	$4p$	$4^k p$	np
Additions	$3(b+p)$	$2b+3p$	$2^k b + 3p(2^k-1)$	$2b+(p-1)n$
Multiplications	$4a$	$2a$	$2^k a$	$2a$

3 CONCLUSION

In this paper, we have proposed a fast block circulant jacket matrices of order $N = np$. on one hand, since the transform is jacket matrices, the inverse transform is easily obtained by the reciprocal and transpose operations.Furthre, it has a fast efficient algorithm. On the other hand, since it is block circulant matrix, it may be applied to the study of the covariance matrices and its factors of linear models of time random processes. In signal processing theory and information theory, there is an important class of random progress. The conventional studies is the generalization of the Gray paper to block Toeplitz matrices, which is a type of matrices frequently used in Communications, Information Theory and Signal Processing, because, for instance, matrix representations of discrete-time causal finite impulse response (FIR) multiple-input multiple-output (MIMO) filters and correlation matrices of vector wide sense stationary (WSS) process and block Toeplitz. Therefore, the conventional MIMO system deals with the asymptotic behaviour of eigenvalues, products and inverse of block Toepliz matrices. The simulation result shows the significance of proposed distributed-multi relay multi-hop system.

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