*Fourteenth International Workshop on Algebraic and Combinatorial Coding Theory*
*September 7–13, 2014, Svetlogorsk (Kaliningrad region), Russia* *pp. 221–226*

# On the Extendability of Quasidivisible Optimal Arcs [1]

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**Abstract.** We prove a new sufficient condition for the extendability of Griesmer arcs with certain parameters.

## 1 Preliminaries

It is a well-known fact that adding a parity check bit to all words of a binary $[n, k, d]$-code with $d$ odd increases the minimum distance, i.e. the resulting code has parameters $[n + 1, k, d + 1]$. This result has been generalized by Hill and Lizak in [4, 5]. They showed that if all weights in an $[n, k, d]_q$ code are congruent to 0 or $d \pmod q$, with $(d, q) = 1$, then it can be extended to an $[n+1, k, d+1]_q$ code. It turned out that this fact has a natural explanation in terms of blocking sets with respect to hyperplanes. It was proved in [6, 9] that the result of Hill and Lizak can be obtained from the well-known Bose-Burton theorem for blocking sets in $\mathrm{PG}(k-1, q)$. A further generalization along these lines was done in [7] by exploiting a result of Beutelspacher and Heim on the size of the minimal non-trivial (i.e. not containing a hyperplane) blocking set in a finite projective geometry.

In a series of papers, Maruta obtained various results [9–11] on the extendability of linear codes. He introduced the notion diversity of a linear code with spectrum $(A_i)$ as the pair $(\Phi_0, \Phi_1)$, where

$$\Phi_0 = \frac{1}{q-1} \sum_{q|i, i\neq 0} A_i, \quad \Phi_1 = \frac{1}{q-1} \sum_{i\not\equiv 0, d(q)} A_i,$$

[1] The first two authors gratefully acknowledge the financial support by the Scientific Research Fund of Sofia University under Contract 37/28.04.2014.

and proved that for various values of the diversity the code is indeed extendable. In particular, he showed that a linear $[n,k,d]$ code over $\mathbb{F}_q$, with $q \geq 5$ odd, having all non-zero weights congruent to $-2,-1$, and 0 modulo $q$ is extendable.

In this note, we introduce the notion of $t$-quasidivisibility modulo $q$. We define a special arc $\widetilde{\mathcal{K}}$ in the dual geometry $\widetilde{\mathrm{PG}}(k-1,q)$ and relate the extendability property for $\mathcal{K}$ with the existence of a hyperplane in the support of $\widetilde{\mathcal{K}}$. The main theorem in this note states that every $t$-quasidivisible Griesmer arc with divisor $q$, $t < \sqrt{q}$, and an additional numerical condition on the parameters, is $t$-extendable.

## 2 Basic definitions

Let $\mathcal{P}$ be the set of points of the projective geometry $\mathrm{PG}(k-1,q)$. Every mapping $\mathcal{K} : \mathcal{P} \to \mathbb{N}_0$ from the pointset of the geometry to the non-negative integers is called a multiset in $\mathrm{PG}(k-1,q)$. This mapping is extended additively to the subsets of $\mathcal{P}$: for every $\mathcal{Q} \subseteq \mathcal{P}$, $\mathcal{K}(\mathcal{Q}) = \sum_{P\in\mathcal{Q}} \mathcal{K}(P)$. The integer $n := \mathcal{K}(\mathcal{P})$ is called the cardinality of $\mathcal{K}$. For every set of points $\mathcal{Q} \subset \mathcal{P}$ we define its characteristic (multi)set $\chi_{\mathcal{Q}}$ by

$$\chi_{\mathcal{Q}}(P) = \begin{cases} 1 & \text{if } P \in \mathcal{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Multisets can be viewed as arcs or as blocking sets. A multiset $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called an $(n,w)$-multiarc (or simply $(n,w)$-arc) if (1) $\mathcal{K}(\mathcal{P}) = n$, (2) $\mathcal{K}(H) \leq w$ for every hyperplane $H$, and (3) there exists a hyperplane $H_0$ with $\mathcal{K}(H_0) = w$. Similarly, a multiset $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called an $(n,w)$-blocking set with respect to the hyperplanes (or $(n,w)$-minihyper) if (1) $\mathcal{K}(\mathcal{P}) = n$, (2) $\mathcal{K}(H) \geq w$ for every hyperplane $H$, and (3) there exists a hyperplane $H_0$ with $\mathcal{K}(H_0) = w$.

An $(n,w)$-arc $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called $t$-extendable, if there exists an $(n+t,w)$-arc $\mathcal{K}'$ in $\mathrm{PG}(k-1,q)$ with $\mathcal{K}'(P) \geq \mathcal{K}(P)$ for every point $P \in \mathcal{P}$. Arcs that are 1-extendable are called simply extendable arcs. An arc is said to be complete if it is not extendable. Similarly, an $(n,w)$-blocking set $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called reducible, if there exists an $(n-1,w)$-blocking set $\mathcal{K}'$ in $\mathrm{PG}(k-1,q)$ with $\mathcal{K}'(P) \leq \mathcal{K}(P)$ for every point $P \in \mathcal{P}$. A blocking set is called irreducible if it is not reducible.

Given a multiset $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$, we denote by $a_i$ the number of hyperplanes $H$ with $\mathcal{K}(H) = i$. The sequence $(a_i)$ is called the spectrum of $\mathcal{K}$. A multiset $\mathcal{K}$ with $\mathcal{K}(\mathcal{P}) = n$ and spectrum $(a_i)$ is said to be divisible with divisor $\Delta > 1$ if $a_i = 0$ for all $i \not\equiv n \pmod{\Delta}$. $\mathcal{K}$ is called $t$-quasidivisible with divisor $\Delta > 1$ (or $t$-quasidivisible modulo $\Delta$) if $a_i = 0$ for all $i \not\equiv n, n+1, \ldots, n+t \pmod{\Delta}$. An easy corollary of Hill and Lizak's result is that every 1-quasidivisible arc with divisor $q$ is extendable. Maruta's theorem from [11] claims that for $q$ odd every 2-quasidivisible arc with divisor $q$ is extendable.

## 3 Extendability of Griesmer Arcs

There exists an one-to-one correspondence between the classes of isomorphic $[n,k,d]_q$ codes and the classes of projectively equivalent $(n, n-d)$-arcs in $\mathrm{PG}(k-1,q)$ [3]. With every multarc $\mathcal{K}$ we can associate many isomorphic linear codes. Fix one of them and denote it by $C_{\mathcal{K}}$. If $C_{\mathcal{K}}$ is a Griesmer code then we call $\mathcal{K}$ a Griesmer arc.

Let $\mathcal{K}$ be a $t$-quasidivisible $(n,w)$-arc with divisor $q$ in $\Sigma = \mathrm{PG}(k-1,q)$, $t < q$. Set $d = n - w$. This is a typical situation when one investigates the existence problem for Griesmer arcs with given parameters.

Define a new multiset $\widetilde{\mathcal{K}}$ in the dual geometry $\widetilde{\Sigma}$ by

$$\widetilde{\mathcal{K}} : \begin{cases} \mathcal{H} & \to \quad \mathbb{N}_0 \\ H & \to \quad \widetilde{\mathcal{K}}(H) = n + t - \mathcal{K}(H) \pmod{q}. \end{cases} \tag{1}$$

In other words, hyperplanes of multiplicity congruent to $n + a \pmod{q}$ become $(t-a)$-points in the dual geometry. The following result is straightforward.

**Theorem 1.** *Let $\mathcal{K}$ be an $(n,w)$-arc in $\Sigma = \mathrm{PG}(k-1,q)$ that is $t$-quasidivisible modulo $q$ with $t < q$. Let $\widetilde{\mathcal{K}}$ be defined by (1). If*

$$\widetilde{\mathcal{K}} = \sum_{i=1}^{c} \chi_{\widetilde{H}_i} + \widetilde{\mathcal{K}}'$$

*for some multiset $\widetilde{\mathcal{K}}'$ and $c$ not necessarily different hyperplanes $\widetilde{H}_1, \ldots, \widetilde{H}_c$ in the dual geometry $\widetilde{\Sigma}$, then $\mathcal{K}$ is $c$-extendable. In particular, if $\widetilde{\mathcal{K}}$ contains a hyperplane in its support then $\mathcal{K}$ is extendable.*

By Theorem 1, the extendability of $t$-quasidivisible arcs is linked with the structure of the multiset $\widetilde{\mathcal{K}}$ defined in the dual geometry. It turns out that this multiset is highly divisible.

**Theorem 2.** *Let $\widetilde{S}$ be subspace of $\widetilde{\Sigma}$ of dimension at least 1. Then*

$$\widetilde{\mathcal{K}}(\widetilde{S}) \equiv t \pmod{q}.$$

By this theorem, the multiset $\widetilde{\mathcal{K}}$ has the following properties:

- the multiplicity of each point is at most $t$;
- the multiplicity of each subspace of dimension $r$, where $1 \le r \le k-1$, is at least $tv_r$.

Here we use the conventional notation $v_r = (q^r - 1)/(q-1)$. Let us note that in the general case we do not know the cardinality of $\widetilde{\mathcal{K}}$.

## 4 A Theorem on the Extendability of Griesmer Arcs

Consider a Griesmer $t$-quasidivisible arc $\mathcal{K}$, $t < q$, with parameters $(n, w)$ in $\mathrm{PG}(k-1, q)$. Set $d = n - w$ and let $C_{\mathcal{K}}$ be a linear code associated with $\mathcal{K}$. The code $C_{\mathcal{K}}$ has parameters $[n, k, d]_q$, where $d$ can be written as

$$d = sq^{k-1} - \sum_{i=0}^{k-2} \varepsilon_i q^i, \quad 0 \leq \varepsilon_i < q. \tag{2}$$

Then we have $\lceil d/q^j \rceil = sq^{k-j-1} - \sum_{i=j}^{k-2} \varepsilon_i q^i$, which implies

$$n = sv_k - \sum_{i=0}^{k-2} \varepsilon_i v_{i+1}. \tag{3}$$

Let us note that with this notation $t = \varepsilon_0$ since $n + \varepsilon_0 \equiv w_1 \pmod{q}$. Denote by $w_j$ the the maximal multiplicity of a subspace $S$ of codimension $j$ of $\mathrm{PG}(k-1, q)$: $w_j = \max_{\operatorname{codim} S = j} \mathcal{K}(S)$, $j = 1, \ldots, k-1$. We have

$$w_j = \sum_{i=j}^{k-1} \lceil \frac{d}{q^i} \rceil = sv_{k-j} - \sum_{i=j}^{k-2} \varepsilon_i v_{i-j}. \tag{4}$$

By convetion $w_0 = n$.

In the next lemmas we establish some important properties of the arc $\widetilde{\mathcal{K}}$. The proofs of the first two lemmas use simple counting arguments.

**Lemma 1.** *Let $\mathcal{K}$ be a $t$-quasidivisible $(n, n-d)$-Griesmer arc with $d$ given by (2). Let $S$ be a subspace of codimension 2 contained in the hyperplane $H_0$ with $\mathcal{K}(H_0) = w_1 - aq$ where $a \geq 0$ is an integer.*

*(i) If $\mathcal{K}(S) = w_2 - a - b$, $0 \leq b \leq t-2$, then $\widetilde{\mathcal{K}}(\widetilde{S}) \leq t + bq$;*

*(ii) If $\mathcal{K}(S) = w_2 - a - b$, $b \geq t-1$, then $\widetilde{\mathcal{K}}(\widetilde{S}) \leq t + (t-1)q$.*

**Lemma 2.** *Let $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ be as in Lemma 1. Let $T$ be a subspace of codimension 3 in $\mathrm{PG}(k-1, q)$ with $\mathcal{K}(T) = w_3$. Then*

$$\widetilde{\mathcal{K}}(\widetilde{T}) \leq t(q+1) + \varepsilon_1 q.$$

The next lemma shows that for certain parameters the arc $\widetilde{\mathcal{K}}(\widetilde{T})$ is minimal, i.e. a sum of lines.

**Lemma 3.** *Let $\mathcal{K}$ be $t$-quasidivisible Griesmer $(n, w)$-arc in $\mathrm{PG}(k-1, q)$, $q \geq 3$ with $d = n - w$ given by (2). Let $\widetilde{\mathcal{K}}$ be defined by (1). Let further $\varepsilon_0, \varepsilon_1 \leq \sqrt{q}$. For every maximal subspace $T$ of codimension 3 in $\mathrm{PG}(k-1, q)$, i.e. a subspace with $\mathcal{K}(T) = w_3$, it holds*

$$\widetilde{\mathcal{K}}(\widetilde{T}) = t(q+1).$$

*Proof.* We have that $\widetilde{T}$ is a plane in $\widetilde{\mathrm{PG}}(k-1,q)$. By Lemma 2, $\widetilde{\mathcal{K}}(\widetilde{T}) \leq \varepsilon_0(q+1)+\varepsilon_1 q$. Set $\widetilde{\mathcal{K}}(\widetilde{T}) = \varepsilon_0(q+1)+\varepsilon_1' q$, where $0 \leq \varepsilon_1' \leq \varepsilon_1$.

Assume $\varepsilon_1' > 0$. Set $\widetilde{\mathcal{F}} = \widetilde{\mathcal{K}}|_{\widetilde{T}}$, i.e. $\widetilde{\mathcal{F}}$ is the restriction of $\widetilde{\mathcal{K}}$ to the plane $\widetilde{T}$ in the dual geometry. Define a dual plane arc $\mathcal{F}$ to $\widetilde{\mathcal{F}}$ by

$$\mathcal{F}(\widetilde{L}) = i \quad \text{iff } \widetilde{\mathcal{F}}(L) = t + iq.$$

Denote by $(A_i)$ the spectrum of $\widetilde{\mathcal{F}}$. We have

$$\begin{aligned} \sum A_{t+iq} &= q^2+q+1 \\ \sum (t+iq)A_{t+iq} &= \varepsilon(q+1)+\varepsilon_1' q \end{aligned}$$

for some $\varepsilon_1' \leq \varepsilon_1$. This implies $\sum_i A_{t+iq} = \varepsilon_1'(q+1)+\varepsilon_0$.

Now let us denote by $B_i$ the number of lines $L$ with $\widetilde{\mathcal{F}}(L) = t+iq$ through a fixed point $P$ of multiplicity $c \geq 0$. Then

$$\begin{aligned} \sum B_{t+iq} &= q+1 \\ \sum (t+iq)B_{t+iq} &= (q+1)\varepsilon_0 + \varepsilon_1' q + cq \end{aligned}$$

which implies $\sum iB_{t+iq} = \varepsilon_1' + c$. Hence $\mathcal{F}$ is a $(\varepsilon_1'(q+1)+\varepsilon_0, \varepsilon_1')$-blocking set.

From $\varepsilon_0, \varepsilon_1 < \sqrt{q}$ and $q \geq 3$ we get that $\varepsilon_0 + \varepsilon_1' < \sqrt{\varepsilon_1' q} + 1$ and, consequently, $\varepsilon'(q+1)+\varepsilon_0 < \varepsilon' q + \sqrt{\varepsilon' q} + 1$. By a well known result Ball [1] and De Beule-Storme-Metsch [2], $\mathcal{F}$ contains a line. Going back to $\widetilde{\mathcal{F}}$ this implies that all lines $L_i$ in $\widetilde{T}$ through $\widetilde{P}$ have multiplicity at least $t+q = \varepsilon_0 + q$. Now we have

$$\begin{aligned} \varepsilon_0(q+1)+\varepsilon_1 q \geq \widetilde{\mathcal{K}}(\widetilde{T}) &= \sum_{i=0}^{q} \widetilde{\mathcal{K}}(L_i) - q\widetilde{\mathcal{K}}(\widetilde{P}) \\ &\geq (q+1)(\varepsilon_0+q) - q\widetilde{\mathcal{K}}(\widetilde{P}) \\ &\geq \varepsilon_0(q+1) + q(q+1) - \varepsilon_0 q. \end{aligned}$$

This implies $q+1 \leq \varepsilon_0 + \varepsilon_1 < 2\sqrt{q}$, i.e. $(\sqrt{q}-1)^2 < 0$, which is a contradiction. Therefore $\varepsilon_1' = 0$ which proves the lemma. □

Using induction on the dimension, we can extend Lemma 3 to subspaces of arbitrary dimension.

**Lemma 4.** *Let $\mathcal{K}$ be $t$-quasidivisible Griesmer $(n,w)$-arc in* $\mathrm{PG}(k-1,q)$, $q \geq 3$ *with $d = n - w$ given by (2). Let $\widetilde{\mathcal{K}}$ be defined by (1). Let $U$ be a subspace in* $\mathrm{PG}(k-1,q)$ *of maximal multiplicity with* $\operatorname{codim} U = r$, $1 \leq r \leq k$ *(if* $\operatorname{codim} U = k$, $U = \varnothing$*). If $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{r-2} < \sqrt{q}$, then*

$$\widetilde{\mathcal{K}}(\widetilde{U}) = \varepsilon_0 v_{r-1}.$$

Now our main theorem becomes almost straightforward.

**Theorem 3.** *Let $\mathcal{K}$ be a $t$-quasidivisible Griesmer $(n, n-d)$-arc with $d$ given by (2). Let $\varepsilon_0 = t, \dots \varepsilon_{k-2} < \sqrt{q}$. Then $\mathcal{K}$ is $t$-extendable.*

*Proof.* By Lemma 4, $\widetilde{\mathcal{K}}$ is a is a $(tv_{k-1}, tv_{k-2})$-minihyper. By corollary 3.5 from [8], every $(tv_{k-1}, tv_{k-2})$ minihyper in $\mathrm{PG}(k-1, q)$ with $t \leq q - \frac{q}{p}$ is the sum of hyperplanes. Now the result follows from Theorem 1 since $t < \sqrt{q} < q - \frac{q}{p}$. □

## References

[1] S. Ball, Multiple blocking sets and arcs in finite planes, *J. London Math. Soc.* (2) **54**(1996), 581–593.

[2] J. De Beule, K. Metsch, L. Storme, Characterization results on weighted minihypers and on linear codes meeting the Griesmer bound, *Adv. in Math. of Comm.* **2**(3)(2008), 261–272.

[3] S. Dodunekov, J. Simonis, Codes and projective multisets, *Electronic Journal of Combinatorics* **5**(1998), R37.

[4] R. Hill, An extension theorem for linear codes, *Des. Codes and Crypt.* **17**(1999), 151–157.

[5] R. Hill, P. Lizak, Extensions of linear codes, in: Proc. Int. Symp. on Inf. Theory, Whistler, BC, Canada 1995.

[6] I. Landjev, The geometric approach to linear codes, in: Finite Geometries, (eds. A. Blokhuis et al.), Kluwer Acad. Publ. 2001, 247–256.

[7] I. Landjev, A. Rousseva, An extension theorem for arcs and linear codes, *Probl. Inform. Transmission* **42**(2006), 65–76.

[8] I. Landjev, P. Vandendriessche, A study of $(xv_t, xv_{t-1})$-minihypers in $\mathrm{PG}(t, q)$, *J. Comb. Theory* Ser. A 119(2012), 1123-1131.

[9] T. Maruta, On the extendability of linear codes *Finite Fields and Appl.* **7**(2001),350–354.

[10] T. Maruta, Extendability of linear codes over $\mathrm{GF}(q)$ with minimum distance, $d$, $\gcd(d, q) = 1$, *Discrete Math.* **266**(2003), 377–385.

[11] T. Maruta, A new extension theorem for linear codes, *Finite Fields and Appl.* **10**(2004), 674–685.