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Covering of F_3^n with spheres of maximal radius

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Abstract. In this paper we consider the problem of covering the space F_3^n with spheres of maximal radius. We present a general approach of finding optimal coverings and provide combinatorial proofs of known computer based results.

1 Introduction

We consider the problem of covering the space F_3^n with spheres of radius n. Equivalently, we want to find a ternary code C with the following property: for any $\mathbf{y} \in F_3^n$ there exists $\mathbf{x} \in C$ such that $\mathbf{d}(\mathbf{x}, \mathbf{y}) = n$ (here $\mathbf{d}(\mathbf{x}, \mathbf{y})$ denotes the usual Hamming distance). The minimum cardinality of such a code is denoted by $T(n)$ and if $|C| = T(n)$ the code C is called optimal.

The sequence $T(n)$ is a part of The on-line encyclopedia of integer sequences, number A086676, [7].

The known results for $T(n)$ for $1 \leq n \leq 13$ are given in Table 1.

n	T(n)		
1	2	7	29
2	3	8	44
3	5	9	68
4	8	10	102–104
5	12	11	153–172
6	18	12	230–264
		13	345–408

Table 1.

For more information for the history of the problem the reader is referred to [1], [2], [3], [4], [5], [6].

It is known that for every n, $1 \leq n \leq 7$ up to equivalence there exists unique optimal covering of F_3^n [1] and for $n = 8$ there exist two optimal coverings [2]. The value $T(9) = 68$ has been found in [3]. The proofs of all results stated above use combinatorial arguments and at some point rely on computer search.

A straightforward recursive bound on $T(n)$ is given by

$$T(n+1) \geq \left\lceil \frac{3}{2} T(n) \right\rceil . \tag{1}$$

In this paper we prove some general results concerning the function $T(n)$. We prove that when (1) is fulfilled with equality the minimum distance of an optimal covering of F_3^{n+1} is one more than the minimum distance of an optimal covering of F_3^n.

Also, we show that for a certain structure of an optimal covering of F_3^n the inequality (1) is strict.

Assume we know the exact value of $T(n)$ and all optimal coverings of F_3^n. Using the two observations outlined above we present a general approach for deciding whether $T(n+1) = \lceil \frac{3}{2} T(n) \rceil$ or $T(n+1) = \lceil \frac{3}{2} T(n) \rceil + 1$ and if so we may find all optimal coverings of F_3^{n+1}.

As a result we present combinatorial proof for finding the values of $T(n)$ for $2 \leq n \leq 7$. Also, we show that for every n, $2 \leq n \leq 6$ there exists a unique optimal covering of F_3^n.

2 General results

Let C be a covering of F_3^{n+1}. For any $i \in \{0,1,2\}$ and k, $1 \leq k \leq n+1$ denote by C_i^k the set of all codewords from C having i in k-th coordinate without this coordinate. It is clear that for $i, j \in \{0,1,2\}$, $i \neq j$ and for any $k, 1 \leq k \leq n+1$ the set $C_i^k \cup C_j^k$ is a covering of F_3^n.

Also, let $c_i^k = |C_i^k|$ and for simplicity when the upper index is missing assume it equals 1.

Lemma 1. *If $T(n+1) = \frac{3}{2} T(n)$ and the minimum distance of all optimal coverings of F_3^n equals t then the minimum distance of all optimal covering of F_3^{n+1} equals $t+1$.*

Proof. Consider an optimal covering C of F_3^{n+1}. It follows from $T(n+1) = \frac{3}{2} T(n)$ that for $i, j \in \{0,1,2\}$, $i \neq j$ and for any $k, 1 \leq k \leq n+1$ the set $C_i^k \cup C_j^k$ is an optimal covering of F_3^n.

Suppose there exist two codeword $u, v \in C$ with $d(u,v) \leq t$. Let $u = (u_1, u_2, \ldots, u_{n+1})$, $v = (v_1, v_2, \ldots, v_{n+1})$ and without loss of generality $u_1 = 0, \neq v_1 = 1$. Thus, for $u' = (u_2, \ldots, u_{n+1})$ and $v' = (v_2, \ldots, v_{n+1})$ we have $d(u', v') = d(u,v) - 1 \leq t - 1$. On the other hand $u', v' \in C_0 \cup C_1$ implying that u' and v' are elements of an optimal covering of F_3^n, thus $d(u', v') \geq t$, a contradiction. Therefore the minimum distance of C is at least $t+1$.

It remains to show that there exist codewords from C at distance $t+1$ apart. Since $C_0 \cup C_1$ is an optimal covering of F_3^n there exist $u', v' \in C_0 \cup C_1$ such that

$d(u', v') = t$. If $u', v' \in C_0$ or $u', v' \in C_1$ then the minimum distance of C is at most t, a contradiction. Therefore $u' \in C_0, v' \in C_1$ or $u' \in C_1, v' \in C_0$ and the minimum distance of C equals $t + 1$. □

Consider an optimal covering C of F_3^n. For any k, $1 \le k \le n$ denote by a_k the number of unordered pairs (u, v), $u, v \in C$ such that $d(u, v) = k$. The set $\{a_1, a_2, \ldots, a_n\}$ is referred to as pair distance distribution of C.

For each k, $1 \le k \le n$ consider a graph G_k with vertices the codewords of C. Two vertices u and v are connected with an edge if and only if $d(u, v) = k$. Call this graph induced graph of C of weight k.

Lemma 2. *Suppose $T(n)$ is even and there exists a unique optimal covering of F_3^n with pair distance distribution $\{a_1, a_2, \ldots, a_n\}$. If there exists k such that $a_k \neq 0$, $a_{k-1} = 0$ and the induced graph G_k has an odd cycle then $T(n+1) > \frac{3}{2}T(n)$.*

Proof. Let $T(n) = 2t$ and assume $T(n+1) = \frac{3}{2}T(n) = 3t$. Let C be an optimal covering of F_3^{n+1}. Since for any $i, j \in \{0, 1, 2\}$, $i \neq j$ the set $C_i \cup C_j$ is a covering of F_3^n we have $c_i + c_j \ge 2t$. Therefore $c_0 = c_1 = c_2 = t$ and the set $C_0 \cup C_1$ is an optimal covering of F_3^n.

We prove that if $u, v \in C_0 \cup C_1$ are such that $d(u, v) = k$ then $u \in C_0, v \in C_1$ or $u \in C_1, v \in C_0$. Indeed, assume $u, v \in C_i$ for $i = 0$ or 1 and let $u = (u_2, \ldots, u_{n+1})$, $v = (v_2, \ldots, v_{n+1})$. Without loss of generality assume $u_2 = 0$ and $v_2 = 1$. Since $C_0^2 \cup C_1^2$ is equivalent to the unique optimal covering of F_3^n, $u' = (i, u_3, \ldots, u_{n+1}), v' = (i, v_3, \ldots, v_{n+1}) \in C_0^2 \cup C_1^2$ and $d(u', v') = d(u, v) - 1 = k - 1$ we get a contradiction with $a_{k-1} = 0$.

Hence, if two vertices u and v of G_k are connected with an edge then $u \in C_0, v \in C_1$ or $u \in C_1, v \in C_0$. This is impossible for the elements of an odd cycle in G_k, a contradiction. Therefore $T(n+1) > 3t = \frac{3}{2}T(n)$. □

3 Main results

In this section we give combinatorial prove of $T(2) = 3$, $T(3) = 5$, $T(4) = 8$, $T(5) = 12$, $T(6) = 18$ and $T(9) = 29$. Also, we show that for every for n, $2 \le n \le 6$ there exists a unique optimal covering of F_3^n.

Lemma 3. *It is true that: (i) $T(2) = 3$; (ii) $T(3) = 5$; (iii) $T(4) = 8$; (iv) $T(5) = 12$; (v) $T(6) = 18$ and for every n, $2 \le n \le 6$ there exists unique optimal covering of F_3^n.*

Proof. (i), (ii) The first two cases $T(2) = 3$ and $T(3) = 5$ are left to the reader. The corresponding unique optimal coverings are given by $\mathcal{C}_2 = \{00, 11, 22\}$ and $\mathcal{C}_3 = \{000, 110, 101, 011, 222\}$. Note that $\mathcal{C}_3$ consists of the four binary vectors of even weight $000, 110, 101, 011$ and a vector 222 of distance 3 from all three of them.

(iii) It follows from $T(3) = 5$ and (1) that $T(4) \geq 8$. Let $\mathcal{C}_4$ be a covering of F_3^4 with cardinality 8. Since $T(3) = 5$ we may assume that $c_0 = c_1 = 3$ and $c_2 = 2$. Therefore both $C_0 \cup C_2$ and $C_1 \cup C_2$ are equivalent to $\mathcal{C}_3$.

Observing the structure of $\mathcal{C}_3$ we conclude that up to equivalence there are two choices for C_2 – $\{000, 222\}$ or $\{000, 011\}$. The corresponding options for C_1 are: $\{110, 101, 011\}$ and $\{110, 101, 222\}$. In the first case there are two possible choices for C_0: $\{110, 101, 011\}$ or $\{112, 121, 211\}$, both do not result in a covering. In the second case there are also two possible choices for C_0: $\{110, 101, 222\}$ or $\{122, 210, 201\}$. The second one gives a covering. Therefore, up to equivalence there exists a unique covering of F_3^4:

$$\{0122, 0210, 0201, 1222, 1110, 1101, 2000, 2011\}.$$

By interchanging $0 \longleftrightarrow 2$ in the second coordinate and $0 \longleftrightarrow 1$ in the third we write this covering as:

$$\mathcal{C}_4 = \{0122, 0000, 0011, 1022, 1100, 1111, 2210, 2201\}.$$

The codewords of $\mathcal{C}_4$ can be partitioned in two sets $A = \{0122, 2201, 1022, 2210\}$ and $B = \{0000, 1100, 1111, 0011\}$ having the following property: The induced graph of weight 4 (respectively 2) of A (respectively B) is a cycle of length 4 and the induced graph of weight 2 (respectively 4) of A (respectively B) consists of two independent edges. Also, the distance between any two codewords from A and B equals 3.

The pair distance distribution of $\mathcal{C}_4$ is given by $a_1 = 0$, $a_2 = 6$, $a_3 = 16$ and $a_4 = 6$.

(iv) It follows from (iii) and (1) that $T(5) \geq 12$. Since there exists a covering of F_3^5 having 12 codewords we conclude that $T(5) = 12$. Let $\mathcal{C}_5$ be a covering of F_3^5 with cardinality 12. According to Lemma 1 the minimum distance of $\mathcal{C}_5$ equals 3.

We have that $c_0 = c_1 = c_2 = 4$ and $C_0 \cup C_1 \equiv C_0 \cup C_2 \equiv C_1 \cup C_2 \equiv \mathcal{C}_4$. Let $C_1 \cup C_2 = \mathcal{C}_4$. Note that if $u, v \in \mathcal{C}_4$ and $d(u, v) = 2$ then $u \in C_1$ and $v \in C_2$ or vice verse. Since each of the two transpositions (12) and (34) is an automorphisms of $\mathcal{C}_5$ we conclude that up to equivalence there is only one choice for C_1 and C_2:

$$C_1 = \{0011, 1100, 1022, 2210\}, C_2 = \{0000, 1111, 0122, 2201\}.$$

Using that $C_0 \cup C_1 \equiv C_0 \cup C_2 \equiv C_1 \cup C_2 \equiv \mathcal{C}_4$ and observing the structure of $\mathcal{C}_4$ it is easy to find that there is only one option for C_0, namely

$$C_0 = \{2020, 1212, 0221, 2102\}.$$

Therefore up to equivalence there exists a unique optimal covering of F_3^5.

For the sets

$$A = \{u_1 = 00000, u_2 = 21120, u_3 = 20211, u_4 = 11202, u_5 = 12021, u_6 = 02112\}$$

and $B = A + \{11111\}$ the code $\mathcal{C}_5$ is equivalent to $A \cup B$. If $B\{v_1, \ldots, v_6\}$ where $v_i = u_i + 11111$ then for $i = 1, 2, \ldots, 6$ we have $d(u_i, v_i) = 5$ and for $i \neq j$ we have $d(u_i, u_j) = 4$, $d(v_i, v_j) = 4$, and $d(u_i, v_j) = 3$. The pair distance distribution of $\mathcal{C}_5$ is $a_3 = 30$, $a_4 = 30$, $a_5 = 6$.

(v) It follows from (iv) and (1) that $T(6) \geq 18$ and since there exists a covering of F_3^6 having 18 codewords we conclude that $T(6) = 18$. Let $\mathcal{C}_6$ be a covering of F_3^6 with cardinality 18. According to Lemma 1 the minimum distance of $\mathcal{C}_6$ equals 4 and since $C_0 \cup C_1 \equiv \mathcal{C}_5$ we easily find that $C_0 \equiv A$ and $C_1 \equiv B$. Observing the structure of $\mathcal{C}_5$ it is easy to find that

$$C_2 = \{22222, 10012, 12100, 00121, 01210, 21001\}.$$

The corresponding covering of F_3^6 is given in the following table:

1.	0 0 0 0 0 0	10.	0 2 0 2 1 1
2.	2 1 2 1 0 0	11.	0 1 2 0 2 1
3.	1 2 2 0 1 0	12.	2 0 0 1 2 1
4.	2 0 1 2 1 0	13.	1 2 0 1 0 2
5.	0 2 1 1 2 0	14.	0 1 1 2 0 2
6.	1 1 0 2 2 0	15.	2 1 0 0 1 2
7.	2 2 1 0 0 1	16.	0 0 2 1 1 2
8.	1 0 2 2 0 1	17.	1 0 1 0 2 2
9.	1 1 1 1 1 1	18.	2 2 2 2 2 2

Optimal covering $\mathcal{C}_6$.

Observe that $\mathcal{C}_6$ consists of 6 sets equivalent to $\{000000, 111111, 222222\}$ and the distance between vectors from distinct sets equals 4. The pair distance distribution of $\mathcal{C}_6$ is given by $a_4 = 135$, $a_6 = 18$. □

Note that for all values of n, $2 \leq n \leq 6$ we have $T(n) = \lceil \frac{3}{2} T(n-1) \rceil$.

Lemma 4. *It is true that* $T(7) = 29$.

Proof. Suppose $T(7) \leq 28$ and consider a covering C of F_3^7 with 28 elements. Since $T(6) = 18$ we have that for any $t = 1, 2, \ldots, 7$ and for any two $i, j \in \{0, 1, 2\}$ it is true that $c_i^t + c_j^t \geq 18$. It follows from $c_0^t + c_1^t + c_2^t = 28$ that for any $t = 1, 2, \ldots, 7$ there exist $i, j \in \{0, 1, 2\}$ such that $c_i^t + c_j^t = 18$. Hence, $C_i^t \cup C_j^t \equiv \mathcal{C}_6$.

Without loss of generality $t = 1, i = 0, j = 1$. Consider three codewords $u = (i, u_2, u_3, \ldots, u_7)$, $v = (i, v_2, v_3, \ldots, v_7)$, $w = (i, w_2, w_3, \ldots, w_7)$ for $i = 0$ or 1. Since $C_0 \cup C_1 \equiv \mathcal{C}_6$ we have that all pairwise distances between u, v, w equal 4 or 6. Assume that for some t we have $\{u_t, v_t, w_t\} = \{0, 1, 2\}$. Without loss of generality $t = 2$. All pairwise distances between the vectors $(i, u_3, \ldots, u_7)$,

$(i, v_3, \ldots, v_7)$, $(i, w_3, \ldots, w_7)$ equal 3 or 6, a contradiction to the fact that two of them are elements of $\mathcal{C}_6$.

Without loss of generality let $000000, 111111 \in C_0$. Since all elements of C_6 contain at least one 2 it follows from the above observations that $C_1 = \mathcal{C}_6 \backslash C_0$. It is obvious that there exist a 0,1,2 coordinate in C_1, a contradiction.

Therefore $T(7) \geq 29$ and since there exists a covering of F_3^7 of cardinality 29, [1] we conclude that $T(7) = 29$. □

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