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The First and Second Most Symmetric Nonsingular Cubic Surfaces

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Abstract. The first and second most symmetric nonsingular cubic surfaces are $x^3 + y^3 + z^3 + t^3 = 0$ and $x^2y + y^2z + z^2t + t^2x = 0$, respectively.

1 Introduction

Throughout this paper k stands for an algebraically closed field of characteristic zero. Let C_d be a nonsingular curve of degree $d \geq 4$ in the two-dimensional projective space. As is well known, its automorphism group $\mathrm{Aut}(C_d)$ is a finite subgroup of $PGL_3(k)$ such that $|\mathrm{Aut}(C_d)| \leq 42d(d-3)$. In fact the maximum value β_d of $|\mathrm{Aut}(C_d)|$ is equal to 168, 360 or $6d^2$, and attained by the Klein quartic curve, the Wiman sextic curve or the Fermat curve according as $d = 4$, $d = 6$ or $d \notin \{4, 6\}$ [5]. Moreover the Klein quartic and the Wiman sextic give rise to highly symmetric MDS codes [2].

Let S_d be a nonsingular surface of degree $d \geq 3$ in the three-dimensional projective space. Its automorphism group $\mathrm{Aut}(S_d)$ is a finite subgroup of $PGL_4(k)$ if $d \neq 4$, may be an infinite group if $d = 4$ [4]. Highly symmetric surfaces may be both of theoretical and of practical interest. Hosoh has completed the classification of the automorphism groups of nonsingular cubic surfaces [1]. According to his classification the maximum and the second maximum of $|\mathrm{Aut}(S_3)|$ is attained by the semidirect product $(\mathbf{Z}_3)^3 \times_s \mathbf{S}_4$ and $\mathbf{S}_5$, respectively.

$M_{m,n}(k)$ stands for the set of all $m \times n$ matrices with entries in k. By definition $M_n(k) = M_{n,n}(k)$, $GL_n(k) = \{A = [a_{ij}] \in M_n(k) \ : \ \det A \neq 0\}$, and $PGL_n(k) = GL_n(k)/(E_n)$, where (E_n) is the subgroup $\{\lambda E_n \ : \ \lambda \in k^*\}$ (E_n is the unit matrix in $GL_n(k)$). The coset $A(E_n)$ containing an $A \in GL_n(k)$ will be denoted (A). We denote by $k[x]$ the k-algebra of polynomials in $x = [x_1, ..., x_n]$ over k. For an $A \in GL_n(k)$ and $f \in k[x]$ we define a polynomial $f_A \in k[x]$ to be $f_A(x) = f(\sum \alpha_{1j}x_j, ..., \sum \alpha_{nj}x_j)$, where $A^{-1} = [\alpha_{ij}]$. As is well known, the map $T_A : k[x] \to k[x]$ assigning f_A to f is a k-algebra isomorphism of $k[x]$ such that $T_A T_B = T_{AB}$, that is, $(f_B)_A = f_{AB}$. $\mathbf{S}_n$ denotes the symmetric group, the group of all permutations of n elements.

A homogeneous polynomial f of degree $d \geq 1$ defines a projective algebraic set $V(f) = \{(a) \in P^{n-1} : f(a) = 0\}$ of an $(n-1)-$dimensional projective space P^{n-1} over k. $V(f)$ is called a hypersurface of degree d. If $n = 4$, $d = 3$, and $V(f)$ is nonsingular, the automorphism group $\mathrm{Aut}(V(f))$ of the hypersurface $V(f)$ is a finite subgroup of $PGL_4(k)$ consisting of $(A) \in PGL_4(k)$ such that $f_A \sim f$ [4]. Let $a = [a_1, ..., a_n] \in k^n$, $(a) \in V(f)$, and $A \in GL_n(k)$. Then (a) is a singular point of $V(f)$ if $f_{x_i}(a) = 0$ for all i. If (a) is a nonsingular point of $V(f)$, $V(\sum_{i=1}^{n} \gamma_i x_i)$ is the tangent plane to $V(f)$ at (a), where $\gamma_i = f_{x_i}(a)$. Clearly $(A) : V(f) \to V(f_A)$ is a bijection, and if $b = Aa$ with $(a) \in V(f)$, then $(f_A)_{x_j}(b) = \sum_{i=1}^{n} f_{x_i}(a)\alpha_{ij}$, where $A^{-1} = [\alpha_{ij}]$. Consequently $(b) = (A)(a)$ is a nonsingular point of $V(f_A)$ if and only if (a) is a nonsingular point of $V(f)$, and the tangent plane of $V(f_A)$ at (b) coincides with $(A)V(\sum_{i=1}^{n} \gamma_i x_i)$. In particular if (a) is a nonsingular point of $V(f)$, $f_A \sim f$ and $(A)(a) = (a)$, then $[f_{x_1}(a), ..., f_{x_n}(a)]A \sim [f_{x_1}(a), ..., f_{x_n}(a)]$.

In §2 it will be shown that the Fermat surface $S = V(x^3 + y^3 + z^3 + t^3)$ is the unique cubic nonsingular surface, up to projective equivalence, such that $\mathrm{Aut}(S)$ is isomorphic to $(\mathbf{Z}_3)^3 \times_s \mathbf{S}_4$. In §3 it will be shown that a surface $S' = V(x^2 t + y^2 z + z^2 t + t^2 x)$ is the unique cubic nonsingular surface such that $\mathrm{Aut}(S')$ is isomorphic to $\mathbf{S}_5$.

2 $\mathbf{Z}_3{}^3 \times_s \mathbf{S}_4$-invariant nonsingular cubic surfaces

Let $\omega \in k^*$ be of order three. Any subgroup of $PGL_4(k)$ isomorphic to $\mathbf{Z}_3$ is conjugate to $\langle(\mathrm{diag}[\omega, 1, 1, 1])\rangle$ or $\langle(\mathrm{diag}[\omega, \omega^2, 1, 1])\rangle$. It can be verified easily that $X = [x_{ij}] \in GL_4(k)$ satisfies $\mathrm{diag}[\omega, 1, 1, 1]X \sim X\mathrm{diag}[\omega, 1, 1, 1]$ (resp. $\mathrm{diag}[\omega, \omega^2, 1, 1]X \sim X\mathrm{diag}[\omega, \omega^2, 1, 1]$) if and only if $x_{1i} = x_{i1} = 0$ for all $i \in [2, 4]$ (resp. $x_{1i} = x_{i1} = x_{2j} = x_{j2} = 0$ for all $i \in [2, 4]$ and all $j \in [3, 4]$). Consequently a subgroup of $PGL_4(k)$ isomorphic to $(\mathbf{Z}_3)^2$ is conjugate to $\langle(A), (B)\rangle$, where $\mathrm{ord}(A) = \mathrm{ord}(B) = 3$ and $(B^j) \in \langle(A)\rangle$ if and only if $j \in 3\mathbf{Z}$. We may assume that $A = \mathrm{diag}[\omega, \omega^2, 1, 1]$ or $A = \mathrm{diag}[\omega, 1, 1, 1]$ and that $B = \mathrm{diag}[b_1, b_2, b_3, b_4]$. Assume first that $A = \mathrm{diag}[\omega, \omega^2, 1, 1]$. If $|\{b_1, b_2, b_3, b_4\}| = 3$, then we may assume that B is equal to one of $S - \{A\}$, where

$$\begin{aligned} S = \{&\mathrm{diag}[1, 1, \omega, \omega^2], \mathrm{diag}[1, \omega, 1, \omega^2], \mathrm{diag}[1, \omega, \omega^2, 1], \mathrm{diag}[\omega, 1, 1, \omega^2], \\ &\mathrm{diag}[\omega, 1, \omega^2, 1], \mathrm{diag}[\omega, \omega^2, 1, 1]\}, \end{aligned}$$

Unless $B = \mathrm{diag}[1, 1, \omega, \omega^2]$, there exist integers i and j such that $(A^i B^j)$ is equal to $(\mathrm{diag}[1, 1, \omega, 1])$ or $(\mathrm{diag}[1, 1, 1, \omega])$. So $\langle(A), (B)\rangle$ is conjugate to $\langle(\mathrm{diag}[\omega, \omega^2, 1, 1]), (\mathrm{diag}[1, 1, \omega, \omega^2])\rangle$ or $\langle(\mathrm{diag}[\omega, \omega^2, 1, 1]), (\mathrm{diag}[1, 1, \omega, 1])\rangle$, provided $|\{b_1, b_2, b_3, b_4\}| = 3$. Clearly $\langle (A), (B)\rangle$ is conjugate to

$$\langle (\mathrm{diag}[\omega, \omega^2, 1, 1]), (\mathrm{diag}[1, 1, \omega, 1)\rangle \quad \text{or} \quad \langle (\mathrm{diag}[\omega, 1, 1, 1]), (\mathrm{diag}[1, 1, \omega, 1])\rangle,$$

provided $|\{b_1, b_2, b_3, b_4\}| = 2$. Assume secondly that $A = \mathrm{diag}[\omega, 1, 1, 1]$. According as $|\{b_1, b_2, b_3, b_4\}|$ is two or three, we may assume B is equal to one of $T - \{A\}$ or S, where

$$T = \{\mathrm{diag}[\omega, 1, 1, 1], \mathrm{diag}[1, \omega, 1, 1], \mathrm{diag}[1, 1, \omega, 1], \mathrm{diag}[1, 1, 1, \omega]\}.$$

Thus a subgroup of $PGL_4(k)$ isomorphic to $(\mathbf{Z}_3)^2$ is conjugate to one of

$$\langle (\mathrm{diag}[\omega, \omega^2, 1, 1]), (\mathrm{diag}[1, 1, \omega, \omega^2)]\rangle, \ \langle (\mathrm{diag}[\omega, \omega^2, 1, 1]), (\mathrm{diag}[1, 1, \omega, 1])\rangle,$$
$$\langle (\mathrm{diag}[\omega, 1, 1, 1]), (\mathrm{diag}[1, \omega, 1, 1])\rangle.$$

Now it is not difficult to verify

Lemma 2.1. A subgroup of $PGL_4(k)$ isomorphic to $(\mathbf{Z}_3)^3$ is conjugate to

$$G_{27} = \langle (\mathrm{diag}[\omega, 1, 1, 1]), (\mathrm{diag}[1, \omega, 1, 1]), (\mathrm{diag}[1, 1, \omega, 1])\rangle.$$

The canonical group representation $\hat{} : \mathbf{S}_4 \to GL_4(k)$ of $\mathbf{S}_4$ is the one such that $\hat{\sigma} x = y$ with $y_i = x_{\sigma^{-1}(i)}$ for any column vector $x \in k^4$. Clearly this representation is an isomorphism. Let $r \geq 2$ be an integer, $\delta \in k^*$ be of order r,

$$D(r) = \{\mathrm{diag}[\lambda_1, \lambda_2, \lambda_3, \lambda_4] \in GL_4(k) \ : \ \lambda_1^r = \lambda_2^r = \lambda_3^r = \lambda_4^r\}.$$

The factor group $(D(r)) = D(r)/k^* E_4$ is isomorphic to

$$\{\mathrm{diag}[\delta^i, \delta^j, \delta^\ell, 1] \ : \ i, j, \ell \in [0, r-1]\}$$

which is isomorphic to $(\mathbf{Z}_r)^3$. A map $\varphi_\sigma : (D(r)) \to (D(r))$ defined by $\varphi_\sigma((A)) = (\hat{\sigma} A \hat{\sigma}^{-1})$ is a group automorphism such that $\varphi_{\sigma\tau} = \varphi_\sigma \circ \varphi_\tau$. The factor group $D(r)\hat{\mathbf{S}}_4/k^* E_4$ is isomorphic to the semidirect product $(D(r)) \times_s \mathbf{S}_4$ such that $((A), \sigma)((A'), \sigma') = ((A)\varphi_\sigma(A'), \sigma\sigma')$. Clearly $(D(r)) \times_s \mathbf{S}_4$ is isomorphic to $(\mathbf{Z}_r)^3 \times_s \mathbf{S}_4$. We note that $\hat{\sigma}\mathrm{diag}[a_1, a_2, a_3, a_4]\hat{\sigma}^{-1} = \mathrm{diag}[b_1, b_2, b_3, b_4]$, where $b_i = a_{\sigma^{-1}(i)}$. We may skip the proof of the following lemma [6].

Lemma 2.2. If $r \geq 3$, then the projective automorphism group of the surface $V(x^r + y^r + z^r + t^r)$ is $D(r)\hat{\mathbf{S}}_4/k^* E_4$, which can be identified with $(D(r)) \times_s \mathbf{S}_4$.

Lemma 2.3. Any $(\mathbf{Z}_3)^3$-invariant nonsingular cubic surface is projectively equivalent to $V(x^3 + y^3 + z^3 + t^3)$. In particular any $(\mathbf{Z}_3)^3 \times_s \mathbf{S}_4$-invariant nonsingular cubic surface is projectively equivalent to $V(x^3 + y^3 + z^3 + t^3)$, and its automorphism group is conjugate to $(D(3)) \times_s \mathbf{S}_4$.

Proof. By Lemma 2.1 it suffices to show that any G_{27}-invariant nonsingular cubic surface is $V(ax^3+by^3+cz^3+dt^3)$, where a, b, c and d are nonzero constants. Let $V(f)$ be a G_{27}-invariant nonsingular cubic surface, where the homogeneous polynomial $f(x,y,z,t)$ of degree three has the form

$$\begin{aligned}&a_1x^3+a_2y^3+a_3z^3+a_4t^3\\&+x^2(b_{12}y+b_{13}z+b_{14}t)+y^2(b_{21}x+b_{23}z+b_{24}t)+z^2(b_{31}x+b_{32}y+b_{34}t)\\&+t^2(b_{41}x+b_{42}y+b_{43}z)+c_1yzt+c_2xzt+c_3xyt+c_4xyz.\end{aligned}$$

Let

$$A_1=\mathrm{diag}[\omega,1,1,1],\ A_2=\mathrm{diag}[1,\omega,1,1],\ A_3=\mathrm{diag}[1,1,\omega,1].$$

It is evident that $\mathrm{ord}(A_i)=3$. Since G_{27} contains (A_i) $(i\in[1,4])$, $f_{A_i^{-1}}$ is equal to one of $\{f,\ \omega f,\ \omega^2 f\}$. If $f_{A_1^{-1}}$ is equal to ωf or $\omega^2 f$, then f is divisible by x, hence $V(f)$ is singular. Assume $f_{A_1^{-1}}=f$. Then $f(x,y,z,t)$ has the form

$$a_1x^3+a_2y^3+a_3z^3+a_4t^3+y^2(b_{23}z+b_{24}t)+z^2(b_{32}y+b_{34}t)+t^2(b_{42}y+b_{43}z)+c_1yzt.$$

Unless $f_{A_2^{-1}}=f$, $V(f)$ is singular. Therefore $f(x,y,z,t)=a_1x^3+a_2y^3+a_3z^3+a_4t^3+b_{34}z^2t+b_{43}t^2z$. Unless $f_{A_3^{-1}}=f$, $V(f)$ is singular. Thus $f(x,y,z,t)=a_1x^3+a_2y^3+a_3z^3+a_4t^3$. Now $V(f)$ is nonsingular if and only if $a_1a_2a_3a_4\neq 0$.

3 $\mathbf{S}_5$-invariant nonsingular cubic surfaces

A subgroup of $PGL_4(k)$ isomorphic to the symmetric group $\mathbf{S}_5$ is one of three groups $C_{5!}$I, $C_{5!}$II, $C_{5!}$III up to conjugacy [3]. We denote these groups by $G(1)$, $G(2)$ and $G(3)$, respectively. There exist group isomorphisms $\varphi_i:\mathbf{S}_5\to G(i)$ such that $(E_1)=\varphi_i((123))$, $(E_2)=\varphi_i((12)(34))$, $(E_3)=\varphi_i((12)(45))$ and $(F)=\varphi_i((12))$ generate $G(i)$ $(i\in[1,3])$ (cf [7, (2.14) in chap.3]). Note that $(123)=(12)(23)$. Let $\omega=\frac{-1+i\sqrt{3}}{2}$. Then $G(1)=\langle(E_1),(E_2),(E_3),(F)\rangle$ (see [3] for the concrete forms of E_1, E_2, E_3 and F).

Lemma 3.1. The $G(1)$-invariant nonsingular cubic surface is $V(f)$, where

$$f(x,y,z,t)=3\sqrt{15}x^3+10(y^3+z^3+t^3)-3\sqrt{15}xy^2-6(\sqrt{15}x+5y)zt.$$

Moreover, $\mathrm{Aut}(V(f))=G(1)$.

Let $\varepsilon=-\frac{\sqrt{5}-1}{4}+i\frac{\sqrt{10+2\sqrt{5}}}{4}$, hence $\mathrm{ord}(\varepsilon)=5$, $\alpha=\frac{-\sqrt{5}+1}{2}$, $\beta=\alpha^2$, $\gamma=-\alpha$, and let

$$H=\begin{bmatrix}\varepsilon^4&0&0&0\\0&\varepsilon^2&0&0\\0&0&\varepsilon&0\\0&0&0&\varepsilon^3\end{bmatrix},\ I=\begin{bmatrix}1&\alpha&\beta&\gamma\\\alpha&\beta&\gamma&1\\\beta&\gamma&1&\alpha\\\gamma&1&\alpha&\beta\end{bmatrix}.$$

Lemma 3.2. (H) and (I) generate in $PGL_4(k)$ a group $G(1)'$ conjugate to $G(1)$. The $G(1)'$-invariant nonsingular cubic surface is $V(x^2y+y^2z+z^2t+t^2x)$. Moreover $\mathrm{Aut}(V(x^2y+y^2z+z^2t+t^2x))=G(1)'$.

Proof. The transpositions $(i\ i{+}1)$ $(i\in[1,4])$ generate $\mathbf{S}_5$, while $(12345)(j\ j{+}1)(12345)^{-1}=(j+1\ j+2)$ $(j\in[1,3])$. Therefore (12345) and (12) generate $\mathbf{S}_5$. Clearly $(123)(34)(45)=(12345)$. In view of the group isomorphism $\varphi_1: S_5\to G(1)$, $\varphi_1((12345))=(E_1FE_2FE_3)$ and $\varphi_1((12))=(F)$. Let

$$K=E_1FE_2FE_3=\begin{bmatrix} -\frac{1}{4} & \frac{\sqrt{15}}{4} & 0 & 0\\ -\frac{\sqrt{15}}{12} & -\frac{1}{12} & \frac{2}{3} & \frac{2}{3}\\ \frac{\sqrt{15}\omega}{6} & \frac{\omega}{6} & \frac{2\omega}{3} & -\frac{\omega}{3}\\ \frac{\sqrt{15}\omega^2}{6} & \frac{\omega^2}{6} & -\frac{\omega^2}{3} & \frac{2\omega^2}{3}\end{bmatrix},\ T=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-1\\0&0&1&1\end{bmatrix}.$$

Now

$$T^{-1}FT=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix},\ T^{-1}KT=\begin{bmatrix} -\frac{1}{4} & \frac{\sqrt{15}}{4} & 0 & 0\\ -\frac{\sqrt{15}}{12} & -\frac{1}{12} & \frac{4}{3} & 0\\ -\frac{\sqrt{15}}{12} & -\frac{1}{12} & -\frac{1}{6} & -i\frac{\sqrt{3}}{2}\\ -i\frac{\sqrt{5}}{4} & -i\frac{\sqrt{3}}{12} & -i\frac{\sqrt{3}}{6} & -\frac{1}{2}\end{bmatrix},$$

$$\det(T^{-1}KT-\lambda E_4)=\lambda^4+\lambda^3+\lambda^2+\lambda+1=\Pi_{i=1}^4(\lambda-\varepsilon^i).$$

If λ is an eigenvalue of $T^{-1}KT$ and $x\in k^4$ satisfies $(T^{-1}KT-\lambda E_4)x=0$, then

$$[x_1,x_2,x_3,x_4]=x_1[1,\ (1+4\lambda)\frac{\sqrt{15}}{15},\ (1+\lambda+3\lambda^2)\frac{\sqrt{15}}{15},\ i(-\lambda^4+\lambda^3)\frac{\sqrt{3}}{5}].$$

So $S^{-1}T^{-1}KTS=\mathrm{diag}[\varepsilon^4,\varepsilon^2,\varepsilon,\varepsilon^3]$, where

$$S=\begin{bmatrix} 1 & 1 & 1 & 1\\ (1+4\varepsilon^4)\frac{\sqrt{15}}{15} & (1+4\varepsilon^2)\frac{\sqrt{15}}{15} & (1+4\varepsilon)\frac{\sqrt{15}}{15} & (1+4\varepsilon^3)\frac{\sqrt{15}}{15}\\ (1+\varepsilon^4+3\varepsilon^3)\frac{\sqrt{15}}{15} & (1+\varepsilon^2+3\varepsilon^4)\frac{\sqrt{15}}{15} & (1+\varepsilon+3\varepsilon^2)\frac{\sqrt{15}}{15} & (1+\varepsilon^3+3\varepsilon)\frac{\sqrt{15}}{15}\\ i(-\varepsilon+\varepsilon^2)\frac{\sqrt{3}}{5} & i(-\varepsilon^3+\varepsilon)\frac{\sqrt{3}}{5} & i(-\varepsilon^4+\varepsilon^3)\frac{\sqrt{3}}{5} & i(-\varepsilon^2+\varepsilon^4)\frac{\sqrt{3}}{5}\end{bmatrix}$$

so that $\det S=-i\frac{8\sqrt{3}}{25}\varepsilon^3(\varepsilon-1)^3(3\varepsilon^3+6\varepsilon^2+4\varepsilon+2)$. We denote the (i,j)-cofactor of $S=[s_{ij}]$ by $\tilde{s}_{ij}$. Note that $S^{-1}\mathrm{diag}[1,1,1,-1]S=E_4+S^{-1}\mathrm{diag}[0,0,0,-2]S$ and that the i-th row of $S^{-1}\mathrm{diag}[0,0,0,-2]S$ is equal to $-2\tilde{s}_{4i}/\det S\times$(4-th row of S). Denote $TS\mathrm{diag}[1,\varepsilon^2,\varepsilon^3,\varepsilon]$ by S'. By computation we can verify $S'^{-1}KS'=H$ and $(S'^{-1}FS')=(I)$.

Finally we shall show that a cubic homogeneous polynomial $f(x,y,z,t)$ of the form as in the proof of Lemma 2.3 such that $f_{H^{-1}}\sim f$, $f_{I^{-1}}\sim f$ and $V(f)$

nonsingular, is proportional to $x^2y+y^2z+z^2t+t^2x$. Since $H^5=E_4$, $f_H^{-1}=\varepsilon^i f$ for some $i\in[0,4]$. If $i=1$, then $f(x,y,z,t)=a_2y^3+b_{14}x^2t+b_31z^3x+c_1yzt$, hence $V(f)$ is singular at $(0,0,0,1)$. Similarly, unless $i=0$, $V(f)$ has a singular point. If $i=0$, then $f(x,y,z,t)=b_1x^2y+b_2y^2z+b_3z^2t+b_4t^2x$. Denoting the coefficients of x^3, y^3, z^3 and t^3 in $f_{I^{-1}}$ by b'_1, b'_2, b'_3 and b'_4, respectively, and observing $\alpha^2=\alpha+1$, we obtain

$$b'=\alpha\begin{bmatrix}1 & \alpha^3 & -\alpha^4 & \alpha\\ \alpha^3 & -\alpha^4 & \alpha & 1\\ -\alpha^4 & \alpha & 1 & \alpha^3\\ \alpha & 1 & \alpha^3 & -\alpha^4\end{bmatrix}b.$$

Since $b'=0$ and the rank of the matrix involved is equal to three, it follows that $b_1=b_2=b_3=b_4\neq 0$. Now let $f=x^2y+y^2z+z^2t+t^2x$. In order to see $f_{I^{-1}}=5(4\alpha+3)f$, we note that denoting $(x+\alpha y+\beta z+\gamma t)^2(x+\alpha x+\beta y+\gamma z+t)$ by $g(x,y,z,t)$, we have $f_{I^{-1}}(x,y,z,t)=g(x,y,z,t)+g(t,x,y,z)+g(z,t,x,y)+g(y,z,t,x)$.

Lemma 3.3. There exist no $G(i)$-invariant nonsingular cubic surfaces for $i\in\{2,3\}$.

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