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# On the construction of optimal codes over $\mathbb{F}_q$ 1

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**Abstract.** In this paper we construct a $q$-divisible $[q^2+q, 5, q^2-q]_q$ code through projective geometry. As the projective dual of the code, we construct optimal codes, giving $n_q(5,d) = g_q(5,d)+1$ for $q^4-q^3-q^2+1 \le d \le q^4-q^3-2q$, $q \ge 3$, where $n_q(k,d)$ is the minimum length $n$ for which an $[n,k,d]_q$ code exists and $g_q(k,d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil$. We also construct a $[g_q(k,d)+1, k, d]_q$ code with $q \ge k \ge 5$ for $(k-2)q^{k-1} - (k-1)q^{k-2} - (q-k+1)q^{k-3} + 1 \le d \le (k-2)q^{k-1} - (k-1)q^{k-2}$.

## 1 Introduction

A linear code $\mathcal{C}$ of length $n$, dimension $k$ and minimum Hamming weight $d$ over the field of $q$ elements $\mathbb{F}_q$ is referred to as an $[n,k,d]_q$ code. The weight distribution of $\mathcal{C}$ is the list of numbers $A_i$ which is the number of codewords of $\mathcal{C}$ with weight $i$. We only consider linear codes having no coordinate which is identically zero. A fundamental problem in coding theory is to find $n_q(k,d)$, the minimum length $n$ for which an $[n,k,d]_q$ code exists ([4]). A natural lower bound on $n_q(k,d)$ is the Griesmer bound: $n_q(k,d) \ge g_q(k,d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil$, where $\lceil x \rceil$ denotes the smallest integer $\ge x$. The values of $n_q(k,d)$ are determined for all $d$ only for some small values of $q$ and $k$, see [11]. In [12], it is proved that there exist no $[g_q(k,d), k, d]_q$ code for $q^{k-1} - q^{k-2} - q^2 + 1 \le d \le q^{k-1} - q^{k-2} - q$ for $k \ge 5$, $q \ge 3$. It is also known for $k \ge 5$, $q \ge 3$ that $[g_q(k,d)+1, k, d]_q$ codes exist for $q^{k-1} - q^{k-2} - 2q + 1 \le d \le q^{k-1} - q^{k-2} - q$, but not known whether such codes exist or not for $q^{k-1} - q^{k-2} - q^2 + 1 \le d \le q^{k-1} - q^{k-2} - 2q$. We note that the part (ii) of Theorem 2.4 in [12] is stated wrongly. The statement should have been $n_q(k,d) \ge g_q(k,d) + 1$ for $s \ge 2$ because the existence of a $[g_q(k,d)+1, k, d]_q$ code is unknown.

**Problem 1.** Does a $[g_q(k,d)+1, k, d]_q$ code exist for $q^{k-1} - q^{k-2} - q^2 + 1 \le d \le q^{k-1} - q^{k-2} - 2q$ for $k \ge 5$, $q \ge 3$?

We give an answer for the case when $k = 5$ as follows:

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**Theorem 1.** *There exists a* $[g_q(5,d)+1,5,d]_q$ *code for* $q^4-q^3-q^2+1 \le d \le q^4-q^3-2q$.

**Corollary 2.** $n_q(5,d) = g_q(5,d)+1$ *for* $q^4-q^3-q^2+1 \le d \le q^4-q^3-2q$.

We construct a $q$-divisible $[q^2+q,5,q^2-q]_q$ code $\mathcal{C}$ through projective geometry. As the projective dual of the code, we construct a $q^2$-divisible $[q^4+1,5,q^4-q^3]_q$ code $\mathcal{C}^*$. And then, we construct $[g_q(k,d)+1,k,d]_q$ codes for $q^4-q^3-q^2+1 \le d \le q^4-q^3-2q$ by (geometric) puncturing.

It is known that $n_q(k,d) = g_q(k,d)$ for all $d \ge (k-2)q^{k-1}-(k-1)q^{k-2}+1$ if $k \ge 3$ and that $n_q(k,d) = g_q(k,d)+1$ for $(k-2)q^{k-1}-(k-1)q^{k-2}-q^2+1 \le d \le (k-2)q^{k-1}-(k-1)q^{k-2}$ if $q \ge 2k-3$ and $k \ge 6$, see [2]. We slightly improve this result. It can be proved applying Theorem 2 in [8] that there exists no $[g_q(k,d),k,d]_q$ code for $(k-2)q^{k-1}-(k-1)q^{k-2}-(k-2)q^{k-4}+1 \le d \le (k-2)q^{k-1}-(k-1)q^{k-2}$ if $q \ge 2k-3$, $k \ge 4$. We show the existence of $[g_q(k,d)+1,k,d]_q$ codes for such $q,k$ and $d$.

**Theorem 3.** *There exists a* $[g_q(k,d)+1,k,d]_q$ *code with* $q \ge k \ge 5$ *for* $(k-2)q^{k-1}-(k-1)q^{k-2}-(q-k+1)q^{k-3}+1 \le d \le (k-2)q^{k-1}-(k-1)q^{k-2}$.

**Corollary 4.** $n_q(k,d) = g_q(k,d)+1$ *for* $(k-2)q^{k-1}-(k-1)q^{k-2}-(k-2)q^{k-4}+1 \le d \le (k-2)q^{k-1}-(k-1)q^{k-2}$ *if* $q \ge 2k-3$, $k \ge 4$.

It is known that $n_q(3,d) = g_q(3,d)+1$ for $q^2-2q-\sqrt{2q}+1 < d \le q^2-2q$ with $q \ge 4$ and that $n_q(k,d) = g_q(k,d)+1$ for $(k-2)q^{k-1}-(k-1)q^{k-2}-2q+1 \le d \le (k-2)q^{k-1}-(k-1)q^{k-2}$ for $q \ge 5$ when $k=4$ and for $q \ge 11$ when $k=5$, see [3] and [9]. See also Corollary 11 in Section 2 for $k=4,5$.

## 2 Construction

We denote by $\mathrm{PG}(r,q)$ the projective geometry of dimension $r$ over $\mathbb{F}_q$. A $j$-flat is a projective subspace of dimension $j$ in $\mathrm{PG}(r,q)$. The 0-flats, 1-flats, 2-flats, 3-flats and $(r-1)$-flats are called *points, lines, planes, solids* and *hyperplanes* respectively. We denote by $\mathcal{F}_j$ the set of $j$-flats of $\mathrm{PG}(r,q)$ and by $\theta_j$ the number of points in a $j$-flat, i.e., $\theta_j = (q^{j+1}-1)/(q-1)$.

Let $\mathcal{C}$ be an $[n,k,d]_q$ code having no coordinate which is identically zero. The columns of a generator matrix of $\mathcal{C}$ can be considered as a multiset of $n$ points in $\Sigma = \mathrm{PG}(k-1,q)$ denoted by $\mathcal{M}_\mathcal{C}$. We see linear codes from this geometrical point of view. An *i-point* is a point of $\Sigma$ which has multiplicity $i$ in $\mathcal{M}_\mathcal{C}$. Denote by $\gamma_0$ the maximum multiplicity of a point from $\Sigma$ in $\mathcal{M}_\mathcal{C}$ and let $C_i$ be the set of $i$-points in $\Sigma$, $0 \le i \le \gamma_0$. We denote by $\Delta_1+\cdots+\Delta_s$ the multiset consisting of the $s$ sets $\Delta_1,\cdots,\Delta_s$ in $\Sigma$. We write $s\Delta$ for $\Delta_1+\cdots+\Delta_s$ when $\Delta_1=\cdots=\Delta_s$. Then, $\mathcal{M}_\mathcal{C} = \sum_{i=1}^{\gamma_0} iC_i$. For any subset $S$ of $\Sigma$, we denote by $\mathcal{M}_{\mathcal{C}}(S)$ the multiset $\{P \in \mathcal{M}_{\mathcal{C}} \mid P \in S\}$. The *multiplicity of S with respect to* $\mathcal{C}$, denoted by $m_{\mathcal{C}}(S)$, is defined as the cardinality of $\mathcal{M}_{\mathcal{C}}(S)$, i.e., $m_{\mathcal{C}}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|$, where $|T|$ denotes the number of elements in a set $T$. Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ such that $n = m_{\mathcal{C}}(\Sigma)$ and $n-d = \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\}$. Such a partition of $\Sigma$ is called an $(n, n-d)$-*arc* of $\Sigma$. Conversely an $(n, n-d)$-arc of $\Sigma$ gives an $[n, k, d]_q$ code in the natural manner. A line $l$ with $t = m_{\mathcal{C}}(l)$ is called a $t$-*line*. A $t$-*plane*, a $t$-*solid* and so on are defined similarly. Denote by $a_i$ the number of $i$-hyperplanes in $\Sigma$. The list of the values $a_i$ is called the *spectrum* of $\mathcal{C}$, which can be calculated from the weight distribution by $a_i = A_{n-i}/(q-1)$ for $0 \le i \le n-d$. An $[n, k, d]_q$ code is called $m$-*divisible* if all codewords have weights divisible by an integer $m > 1$.

**Lemma 5** ([14])**.** *Let $\mathcal{C}$ be an $m$-divisible $[n, k, d]_q$ code with $q = p^h$, $p$ prime, where $m = p^r$ for some $1 \le r < h(k-2)$ satisfying $\lambda_0 > 0$. Then there exists a $t$-divisible $[n^*, k, d^*]_q$ code $\mathcal{C}^*$ with $t = q^{k-2}/m$, $n^* = ntq - \frac{d}{m}\theta_{k-1}$, $d^* = ((n-d)q - n)t$.*

Note that a generator matrix for $\mathcal{C}^*$ is given by considering $(n-d-jm)$-hyperplanes as $j$-points in the dual space $\Sigma^*$ of $\Sigma$ for $0 \le j \le w-1$ [14]. $\mathcal{C}^*$ is called the *projective dual* of $\mathcal{C}$, see also [1] and [5].

**Lemma 6** ([13],[10])**.** *Let $\mathcal{C}$ be an $[n, k, d]_q$ code and let $\cup_{i=0}^{\gamma_0} C_i$ be the partition of $\Sigma = \mathrm{PG}(k-1, q)$ obtained from $\mathcal{C}$. If $\cup_{i \ge 1} C_i$ contains a $t$-flat $\Pi$ and if $d > q^t$, then there exists an $[n - \theta_t, k, d']_q$ code $\mathcal{C}'$ with $d' \ge d - q^t$.*

The code $\mathcal{C}'$ in Lemma 6 can be constructed from $\mathcal{C}$ by removing the $t$-flat $\Pi$ from the multiset for $\mathcal{C}$. We denote the resulting multiset by $\mathcal{C} - \Pi$. In general, the method for constructing new codes from a given $[n, k, d]_q$ code by deleting the coordinates corresponding to some geometric object in $\mathrm{PG}(k-1, q)$ is called *geometric puncturing*, see [10].

Recall that an $[n, k, d]_q$ code $\mathcal{C}$ gives the partition $\bigcup_{i=0}^{\gamma_0} C_i$ of $\Sigma = \mathrm{PG}(k-1, q)$ such that $n = m_{\mathcal{C}}(\Sigma)$ and $n - d = \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\}$. Such a partition of $\Sigma$ is called an $(n, n-d)$-*arc* of $\Sigma$. Conversely an $(n, n-d)$-arc of $\Sigma$ gives an $[n, k, d]_q$ code in the natural manner. A set $S$ of $s$ points in $\mathrm{PG}(r, q)$, $r \ge 2$, is called an $s$-*arc* if no $r+1$ points are on the same hyperplane, see [6] and [7] for arcs. When $q \ge r$, one can take a normal rational curve as a $(q+1)$-arc in $\mathrm{PG}(r, q)$ [[6], Theorem 27.5.1]. A set of $m$ hyperplanes $\mathcal{H}$ in $\Sigma$ is called an *$m$-arc of hyperplanes* if the corresponding set of points forms an $m$-arc in the dual space $\Sigma^*$.

Let $\delta$ be a plane of $\Sigma = \mathrm{PG}(4, q)$. Take a $(q+1)$-arc $K = \{Q_0, Q_1, \cdots, Q_q\}$ in $\delta$ and a line $\ell = \{P_0, P_1, \cdots, P_q\}$ of $\Sigma$ so that $\ell$ and $\delta$ have no common point. Let $l_i$ be the line joining $Q_i$ to $P_i$ for $0 \le i \le q$. Setting $C_1 = (\cup_{i=0}^{q} l_i) \setminus \ell$ and $C_0 = \Sigma \setminus C_1$, we get a $q$-divisible $[q^2+q, 5, q^2-q]_q$ code $\mathcal{C}$.

**Lemma 7.** (1) *There exists a $q$-divisible $[q^2+q, 5, q^2-q]_q$ code $\mathcal{C}$ with spectrum $(a_0, a_q, a_{2q}) = ((q^2-q)/2, q^4-q^2+q+1, (2q^3+3q^2+q)/2)$.*

(2) $\mathcal{C}^*$, *the projective dual of* $\mathcal{C}$, *is a* $q^2$*-divisible* $[q^4+1, 5, q^4-q^3]_q$ *code. The multiset for* $\mathcal{C}^*$ *contains* $q-1$ *mutually disjoint lines.*

*Proof.* (1) The spectrum of $\mathcal{C}$ can be derived as follows. Let $b_i$ be the number of lines in $\delta$ meeting $K$ in exactly $i$ points. Then we have $b_2 = (q^2+q)/2$, $b_1 = q+1$ and $b_0 = (q^2-q)/2$. Let $\pi$ be a solid in $\Sigma = \mathrm{PG}(4,q)$. Assume $\pi$ contains $\ell$. Then $\pi$ is a $2q$-solid, a $q$-solid and a 0-solid if $\pi$ meets $K$ in $\delta$ in a bisecant, a tangent and an external line, respectively. Assume $\pi$ does not contain $\ell$. If $\pi$ contains none of $l_0, l_1, \cdots, l_q$, then $\pi$ is a $q$-solid. If $\pi$ contains $l_0$, then $\pi$ contains none of $l_1, \cdots, l_q$, so, $\pi$ is a $2q$-solid. Thus, $a_0 = b_0$, $a_{2q} = b_2 + q^2(q+1)$, $a_q = \theta_4 - a_0 - a_{2q}$.
(2) It follows from Lemma 5 that $\mathcal{C}$ is a $q^2$-divisible $[q^4+1, 5, q^4-q^3]_q$ code. Let $\ell^*$ and $l_i^*$ be the planes in the dual space $\Sigma^*$ of $\Sigma$ corresponding to $\ell$ and $l_i$ in $\Sigma$, respectively, for $0 \le i \le q$. Let $L_i = \ell^* \cap l_i^*$. Then, $L_i$ is a 1-line in $\ell^*$ for $\mathcal{C}^*$ and $\mathcal{L} = \{L_0, L_1, \cdots, L_q\}$ forms a $(q+1)$-arc of lines in $\ell^*$. Note that every 0-point in $\Sigma^*$ for $\mathcal{C}^*$ is a point on some plane $l_i^*$ or a point in $\ell^*$ on some two lines from $\mathcal{L}$. Let $R_0$ be the 1-point in $L_0$ for $\mathcal{C}^*$. Since any line through $R_0$ meeting none of $l_1^*, \cdots, l_q^*$ and not being contained in $l^* \cup l_0^*$ contains no 1-point, the number of lines through $R_i$ containing a 0-point is at most $(\theta_2 - \theta_1)q + 2q + 1 = q^3 + 2q + 1$. Hence, one can take at least $(\theta_3 - q^3 - 2q - 1)/q = q - 1$ mutually disjoint lines containing no 0-point for $\mathcal{C}^*$. □

From Lemma 7 (2), we can construct a $[q^4+1-t(q+1), t, q^4-q^3-tq]_q$ code for $1 \le t \le q-1$ from our code $\mathcal{C}^*$ by geometric puncturing. This provides the codes needed in Theorem 1 when $d$ is divisible by $q$. The rest of the codes required for the theorem can be obtained by puncturing these divisible codes.

**Remark.** The projective dual of a $q^{k-3}$-divisible $[q^{k-1}+1, k, q^{k-1}-q^{k-2}]_q$ code is a $q$-divisible $[q^2+q, k, q^2-q]_q$ code for $k \ge 4$. For $k=4$, one can construct $q$-divisible $[q^2+q, 4, q^2-q]_q$ code from $q$ skew lines in $\mathrm{PG}(3,q)$. But for $k \ge 6$, the existence of a $q$-divisible $[q^2+q, k, q^2-q]_q$ code is unknown except for the extended ternary Golay code ($k=6$ and $q=3$).

The following result is interpreted from the necessary and sufficient condition for the existence of Griesmer codes of Belov type, see [4], [5].

**Theorem 8** ([4])**.** *For given positive integers* $s$ *and* $u_r \le \cdots \le u_1 < k$ *satisfying* $u_i > u_{i+q-1}$ *for* $1 \le i \le r-q+1$*, there exists a* $(u_j-1)$*-flat* $\Delta_{u_j-1}$ *in* $\Sigma = \mathrm{PG}(k-1,q)$ *for* $1 \le j \le r$ *such that the multiset* $s\Sigma$ *contains the multiset* $\Delta_{u_1-1} + \cdots + \Delta_{u_r-1}$ *if and only if* $\sum_{i=1}^{m} u_i \le sk$*, where* $m = \min\{s+1, r\}$*.*

Note that in the proof of Theorem 2.12 in [4], $A(f_1(x)), \cdots, A(f_k(x))$ with $\deg f_i = 1$ for $1 \le i \le k$ correspond to $k$ distinct hyperplanes whose defining vectors give a $k$-arc in $\mathrm{PG}(k-1,q)$.

For $k=4$, it is known that $n_q(4,d) = g_q(4,d)$ for $d \ge 2q^3-3q^2+1$ for all $q$ and that $n_q(4,d) = g_q(4,d)+1$ for $2q^3-3q^2-q+1 \le d \le q^3-3q^2$ for $q \ge 4$.

**Lemma 9.** *There exists a* $[g_q(4,d)+1,4,d]_q$ *code for* $2q^3-4q^2+1 \le d \le 2q^3-3q^2$ *for any* $q$.

*Proof.* Let $H_1, H_2, H_3$ be three planes in $\Sigma = \mathrm{PG}(3,q)$ such that $H_1 \cap H_2 \cap H_3$ is a point, say $P$. Then the multiset $\mathcal{S} = 2\Sigma + P - (H_1 + H_2 + H_3)$ gives a $[g_q(4,d)+1,4,d]_q$ code for $d = 2q^3 - 3q^2$ and the set of 0-points in the multiset $\mathcal{S}$ consists of three lines through $P$. So, one can take $q-1$ lines $l_1, l_2, \cdots, l_{q-1}$ containing none of the 0-points. Hence, by Lemma 6, the multiset $\mathcal{S} - (l_1 + \cdots + l_t)$ gives a $[g_q(4,d)+1,4,d]_q$ code for $d = 2q^3 - 3q^2 - tq$ for $1 \le t \le q-1$. The other codes required can be obtained by puncturing. □

For $k = 5$, we can prove the following similarly.

**Theorem 10.** *There exists a* $[g_q(5,d)+1,5,d]_q$ *code for* $3q^4 - 5q^3 + 1 \le d \le 3q^4 - 4q^3$ *for any* $q$.

**Corollary 11.** $n_q(k,d) \le g_q(k,d) + 1$ *for any* $q$ *for*
(a) $2q^3 - 4q^2 + 1 \le d \le 2q^3 - 3q^2$ *when* $k = 4$.
(b) $3q^4 - 5q^3 + 1 \le d \le 3q^4 - 4q^3$ *when* $k = 5$.

**Problem 2.** Does a $[g_q(k,d)+1,k,d]_q$ code exist for $(k-2)q^{k-1} - kq^{k-2} + 1 \le d \le (k-2)q^{k-1} - (k-1)q^{k-2}$ for $k \ge 6$?

To prove Theorem 3, it suffices to show the following.

**Lemma 12.** *There exists a* $[g_q(k,d)+1,k,d]_q$ *code with* $q \ge k \ge 5$ *for* $d = (k-2)q^{k-1} - (k-1)q^{k-2} - \sum_{i=1}^{k-3} t_i q^i$ *with* $0 \le t_{k-3} \le q-k$ *and* $0 \le t_j \le q-1$ *for* $1 \le j \le k-4$.

*Proof.* Let $\{H_1, H_2, \cdots, H_k\}$ be a $k$-arc of hyperplanes in $\Sigma = \mathrm{PG}(k-1,q)$, that is, at most $k-1$ hyperplanes of which are on a same point. Then, $H_1 \cap \cdots \cap H_{k-1}$ is a point, say $P$, and $P \notin H_k$. Let $\mathcal{S}$ be the multiset given by the $k-2$ copies of $\Sigma$ plus $P$ with $k-1$ hyperplanes $H_1, \cdots, H_{k-1}$ deleted, i.e., $\mathcal{S} = (k-2)\Sigma + P - (H_1 + \cdots + H_{k-1})$ and let $\mathcal{C}$ be the code given by $\mathcal{S}$. Then $\mathcal{C}$ is a $[g_q(k,d)+1,k,d]_q$ code with $d = (k-2)q^{k-1} - (k-1)q^{k-2}$, and the set of 0-points in $\Sigma$ consists of $k-1$ lines through $P$ meeting $H_k$ in $k-1$ points. Let $\pi_i = H_k \cap H_i$ for $1 \le i \le k-1$. Then, the set $\{\pi_1, \cdots, \pi_{k-1}\}$ forms a $(k-1)$-arc of $(k-3)$-flats in $H_k$ and the multiset $\mathcal{M}_{\mathcal{C}}(H_k)$ can be written as $\mathcal{M}_{\mathcal{C}}(H_k) = (k-2)H_k - (\pi_1 + \cdots + \pi_{k-1})$. Since $(k-1)$-arcs in a $(k-2)$-flat are unique up to projective equivalence, it follows from Theorem 8 that the multiset $\mathcal{M}_{\mathcal{C}}(H_k)$ contains $\Delta_{u_1} + \cdots + \Delta_{u_r}$, where $\Delta_{u_j}$ is a $u_j$-flat in $H_k$ for $1 \le j \le r$ with $u_r \le \cdots \le u_1 < k-2$ such that at most $q-1$ of $u_1, \cdots, u_r$ are the same value and that $\Delta_{u_j} = \pi_j$ for $1 \le j \le k-1$. So, the multiset $\mathcal{M}_{\mathcal{C}}(H_k) - (\Delta_{u_1} + \cdots + \Delta_{u_r})$ gives a $[g_q(k,d)+1,k,d]_q$ code for $d = (k-2)q^{k-1} - (k-1)q^{k-2} - \sum_{i=1}^{r} q^{u_i}$. □

## References

[1] A.E. Brouwer and M. van Eupen, The correspondence between projective codes and 2-weight codes, *Des. Codes Cryptogr.*, **11**, 261–266, 1997.

[2] E.J. Cheon, A class of optimal linear codes of length one above the Griesmer bound, *Des. Codes Cryptogr.*, **51**, 9–20, 2009.

[3] E.J. Cheon, T. Kato and S.J. Kim, Nonexistence of a $[g_q(5, d), 5, d]_q$ code for $3q^4 - 4q^3 - 2q + 1 \leq d \leq 3q^4 - 4q^3 - q$, *Discrete Math.*, **308**, 3082–3089, 2008.

[4] R. Hill, Optimal linear codes, in: Mitchell C. (ed.) *Cryptography and Coding II*, pp. 75–104. Oxford Univ. Press, Oxford, 1992.

[5] R. Hill and E. Kolev, A survey of recent results on optimal linear codes, in: Holroyd F.C. et al (ed.) *Combinatorial Designs and their Applications*, pp.127–152, Chapman and Hall/CRC Press Research Notes in Mathematics. CRC Press. Boca Raton, 1999.

[6] J.W.P. Hirschfeld, *Projective Geometries over Finite Fields*, Second edition, Clarendon Press, Oxford, 1998.

[7] J.W.P. Hirschfeld and J.A. Thas, *General Galois Geometries*, Clarendon Press, Oxford, 1991.

[8] A. Klein and K. Metsch, Parameters for which the Griesmer bound is not sharp, *Discrete Math.*, **307**, 2695–2703, 2007.

[9] K. Kumegawa and T. Maruta, Nonexistence of some Griesmer codes of dimension 4 over $\mathbb{F}_q$, preprint.

[10] T. Maruta, Construction of optimal linear codes by geometric puncturing, *Serdica J. Computing*, **7**, 73–80, 2013.

[11] T. Maruta, Griesmer bound for linear codes over finite fields, `http://www.mi.s.osakafu-u.ac.jp/~maruta/griesmer.htm.`

[12] T. Maruta, I.N. Landjev and A. Rousseva, On the minimum size of some minihypers and related linear codes, *Des. Codes Cryptogr.*, **34**, 5–15, 2005.

[13] T. Maruta and Y. Oya, On optimal ternary linear codes of dimension 6, *Adv. Math. Commun.*, **5**, 505–520, 2011.

[14] M. Takenaka, K. Okamoto and T. Maruta, On optimal non-projective ternary linear codes, *Discrete Math.*, **308**, 842–854, 2008.