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Multiaccess problem with two active users. [1]

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Abstract. We consider the multiaccess problem with two active users. This problem is equivalent to the the classical $(2, N)$ group testing problem, i.e., the problem of finding two defectives among N elements. We propose a new adaptive algorithm such that for $N = \lfloor 2^{\frac{t+1}{2}} - t2^{\frac{t}{4}} \rfloor$ the problem can be solved in t tests.

1 Introduction

Assume that there are N users who transmit their messages in the form of binary sequences of length t. During transmission through a channel, exactly two users can be active (i.e., transmit their messages according to a transmission strategy designed beforehand). At each time instant (from 1 to t), the channel output is 0 if both users transmit zeros at this instant, and in all other cases the channel output is 1. A transmission strategy must be such that, given the channel output, one can find out which users transmitted their messages. As above, if a transmission strategy may change depending on the channel output at each time instant, the strategy is said to be adaptive. We will consider adaptive strategies only.

Let us formulate the problem under consideration in somewhat different terms.

In the classical group testing model, there is a set $[N] := \{1, \dots, N\}$ of elements containing a subset $\mathcal{D} \subset [N]$ of defectives. The main problem of group testing is determining $\mathcal{D}$ in the fewest number of tests. Each test is some subset of $[N]$. It is assumed that there is a test function which for any subset $\mathcal{S} \subset [N]$ indicates the presence of a defective in this subset (gives an answer to the test). Formally, a test function $f_{\mathcal{S}} : 2^{[N]} \to \{0, 1\}$ can be defined as follows:

$$f_S(\mathcal{D}) = \begin{cases} 0 & \text{if} \quad |\mathcal{S} \cap \mathcal{D}| = 0 \\ 1 & \text{if} \quad |\mathcal{S} \cap \mathcal{D}| > 0. \end{cases} \tag{1}$$

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A set of tests forms a search algorithm. We say that a search algorithm is successful if after applying it we can uniquely determine $\mathcal{D}$ from the answers $f_{\mathcal{S}_1}, \ldots, f_{\mathcal{S}_t}$. Algorithms can be adaptive and nonadaptive. In an adaptive algorithm, when choosing a test one can use results of previous tests. In a nonadaptive algorithm, all tests are independent. In this paper we consider adaptive search algorithms only.

Below we need the following notation. Let $|\mathcal{D}| = D$ be the number of defectives, and $N_t(D)$ the largest number of elements among which D defectives can be found in t tests. For an adaptive algorithm $a = a(N, D, t)$, denote by $a_t(D)$ the maximal number of elements for which it is proved that the $D, a_t(D)$ problem can be solved in t tests, i.e., algorithm a is successful. Thus, $a_t(D)$ is a lower bound for $N_t(D)$.

More detailed description of previous results and the present new results can be found in [1]. A description of this and many other interesting combinatorial search and group testing models can be also found in [2] and [3].

In [4], the authors obtained an upper bound $N_t(2) \leq \lfloor 2^{(t+1)/2} - 1/2 \rfloor$ and proved for a search algorithm l proposed by them that $\frac{l_t(2)}{N_t(2)} > 0.95$. This result was improved in [5], where a proposed search algorithm u yielded the bound $\frac{u_t(2)}{N_t(2)} > 0.983$. Moreover, it was also shown in [5] that for a search algorithm v there exists t_0, such that $\frac{v_t(2)}{N_t(2)} > 0.995$ for $t \geq t_0$.

To conclude this section, we present a result which will be used below. It was obtained in [6] for a special case where two defectives are contained in two disjoint subsets of $[N]$, one in each subset.

Lemma 1. *Assume that a set $A \subset [N]$ is known to contain exactly one defective, a set $B \subset [N]$ is also known to contain exactly one defective, and these sets are disjoint. Then the minimal number of tests required to find the defective in A, $|A| = m$, and the defective in B, $|B| = n$, is $\lceil \log mn \rceil$.*

2 Description of the algorithm $w(N, 2, t)$

Consider $w_t = \lfloor 2^{\frac{t+1}{2}} - t2^{\frac{t}{4}} \rfloor$ elements, and let $[N] = \mathcal{Z} \cup \mathcal{X} \cup \mathcal{Y}$ be a partition for which we may claim that $f_{\mathcal{Z}}(\mathcal{D}) = 0$, $f_{\mathcal{X}}(\mathcal{D}) = 1$, and there are no elements in $\mathcal{Y}$ that can be added to either $\mathcal{X}$ or $\mathcal{Z}$ without violating these conditions. Before the first test, we put $\mathcal{X} = [N]$, $\mathcal{Z} = \emptyset$, $\mathcal{Y} = \emptyset$. Let us follow up changes in this partition after each test.

- **Test 1:** For the first test, take $\mathcal{S}_1 = [1, x_1]$, where $x_1 = \lfloor (\sqrt{2}-1)2^{\frac{t}{2}} \rfloor$. If $f_{\mathcal{S}_1} = 0$, we obtain

$$\mathcal{Z} = [1, x_1],\ \mathcal{X} = [x_1+1, w_t],\ \mathcal{Y} = \emptyset,$$

and the problem reduces to the same algorithm but with fewer elements. If $f_{\mathcal{S}_1} = 1$, we obtain $\mathcal{X} = [1, x_1]$, $\mathcal{Y} = [x_1+1, w_t]$, $\mathcal{Z} = \emptyset$. Let us estimate the possible number A_1 of variants of location of the two defectives after this answer to the first test:

$$A_1 = \binom{x_1}{2} + x_1 \cdot (w_t - x_1) \leq 2^{t-1} - t(\sqrt{2}-1)2^{\frac{3t}{4}}$$

- **Test 2 (after the answer $f_{\mathcal{S}_1} = 1$):**

For the second test, take $\mathcal{S}_2 = [1, x_2]$, where x_2 is an integer such that the number

$$A_2 := \binom{x_2}{2} + x_2 \cdot (w_t - x_2)$$

is the nearest to $A_1/2$.

Note that $A_1/2$ need not necessarily be an integer. We take the integer nearest to $A_1/2$. To estimate A_2, compute the difference between the number of possible variants of location of the two defectives after tests of cardinalities $i+1$ and i:

$$\frac{(i+1)i}{2} + (i+1)(w_t - i - 1) - \frac{i(i-1)}{2} - i(w_t - i) = w_t - i - 1 \leq w_t.$$

Hence,

$$A_2 \leq 2^{t-2} - t(\sqrt{2}-1)2^{\frac{3t}{4}-1} + w_t.$$

Note that this estimate for the number of possible variants of location of the two defectives does not depend on what was the answer to the second test. In the case of $f_{\mathcal{S}_2} = 0$ we obtain the partition

$$\mathcal{Z} = [1, x_2],\ \mathcal{X} = [x_2+1, x_1],\ \mathcal{Y} = [x_1+1, w_t],$$

and in the case of $f_{\mathcal{S}_2} = 1$, the partition

$$\mathcal{Z} = \emptyset,\ \mathcal{X} = [1, x_2],\ \mathcal{Y} = [x_2 + 1, w_t].$$

But in both cases the estimate for A_2 is valid, since we divide all variants of location of the two defectives in half (taking due account of integer values).

After that, we proceed similarly. Assume that after k tests we obtain a partition $\mathcal{Z} = [1, z_k]$, $\mathcal{X} = [z_k + 1, z_k + x_k]$, $\mathcal{Y} = [z_k + x_k + 1, w_t]$ (the cardinality of $\mathcal{Y}$ will be denoted by y_k hereafter).

- **Test** $(k+1)$**:** For the $(k+1)$ st test, we take the interval $[z_k+1, z_k+x_{k+1}]$ of length x_{k+1}, where x_{k+1} is an integer such that

$$A_{k+1} = \binom{x_{k+1}}{2} + x_{k+1} \cdot y_{k+1} \tag{2}$$

is the nearest to $A_k/2$.

Recall that $w_t = \lfloor 2^{(t+1)/2} - t2^{\frac{t}{4}} \rfloor$, hence,

$$A_{k+1} \leq 2^{t-k-1} - t(\sqrt{2} - 1)2^{\frac{3t}{4}-k} + kw_t \tag{3}$$

Note that (2) immediately implies

$$x_{k+1} \leq \frac{A_{k+1}}{y_{k+1}}. \tag{4}$$

Thus, after $k + 1$ tests, we obtain the partition

$$\mathcal{Z} = [1, z_{k+1}],\ \mathcal{X} = [z_{k+1} + 1, z_{k+1} + x_{k+1}],\ \mathcal{Y} = [z_{k+1} + x_{k+1} + 1, w_t]$$

(the set $\mathcal{Y}$ contains y_{k+1} elements, and (4) holds).

Tests from the $(k + 2)$nd to the $(2k + 1)$st are aimed at reducing the set $\mathcal{Y}$. A zero answer moves elements from $\mathcal{Y}$ to $\mathcal{Z}$, and answer "1" defines two sets, each of them containing exactly one defective.

- **Test** $(k+2)$**:** For the $(k+2)$nd test, take the first $\lfloor \frac{2^{t-k-2}}{x_{k+1}} \rfloor$ elements of $\mathcal{Y}$ (if there are less elements in $\mathcal{Y}$, take all of them).

  If the answer to this test is "1", then by the lemma we can find two defectives in $2^{\frac{t+1}{2}}$ tests.

  the answer to this test is "0", then we include $\lfloor \frac{2^{t-k-3}}{x_{k+1}} \rfloor$ elements of $\mathcal{Y}$ in the next test, and so on. If at some moment the answer to a test is "1", after that we apply the algorithm of the lemma. If all answers are "0", then tested elements are moved from $\mathcal{Y}$ to $\mathcal{Z}$ and we continue the procedure till there are elements in $\mathcal{Y}$.

- **Test** $(2k+1)$**:** Take $\lfloor \frac{2^{t-2k-1}}{x_{k+1}} \rfloor$ elements of $\mathcal{Y}$ (if there are less elements remaining in $\mathcal{Y}$, take all of them). In [1] we proved that if we take $k = \lfloor \frac{t}{4} \rfloor$, then after the $(2k+1)$st test (or earlier) there will be no elements remaining in $\mathcal{Y}$. Thus, the problem is again reduced to the same algorithm but with fewer elements (the number of remaining elements is x_{k+1}).

3 Main result and table for small N

Next theorem show that $(2, \lfloor 2^{\frac{t+1}{2}} - t2^{\frac{t}{4}} \rfloor)$ problem can be solved for t tests.

Theorem 1. *For the adaptive algorithm $w = w(N, 2, t)$ we have*

$$w_t(2) = \lfloor 2^{\frac{t+1}{2}} - t2^{\frac{t}{4}} \rfloor.$$

Corollary. *As $t \to \infty$, we have*

$$\frac{w_t}{N_t(2)} \to 1.$$

As was already noted, in [5] there was constructed an algorithm solving the $(2, c_t)$ problem in t tests, where

$$c_t = \begin{cases} 89 \cdot 2^{k-6} & \text{for} \quad t = 2k \geq 12, \\ 63 \cdot 2^{k-5} & \text{for} \quad t = 2k+1 \geq 13. \end{cases}$$

In [1] we show that the algorithm can be improved and we have following table

t	16	17	18	19	20
$N_t(2) \geq$	357	506	717	1015	1437

Now we show that using the idea of our algorithm $w(N, 2, t)$ we obtain good results also for some small N.

Theorem 2. *We have*

$$N_{20}(2) \geq 1438.$$

Proof. For the first test, take $\mathcal{S}_1 = [1, 423]$ and after the answer $f_{\mathcal{S}_1} = 1$ we take $\mathcal{S}_2 = [1, 193]$.

- **If** $f_{\mathcal{S}_2} = 1$**:** We ask 679 then 339 then 169 other elements until we have positive answer and then use Lemma 1. If all answer are negative we have $193 + 58 = 251$ elements and 15 answer. But $N_{15}(2) \geq 252$.
- **If** $f_{\mathcal{S}_2} = 0$**:** We ask 569 (then 284, then 142) other elements until we have positive answer and then use Lemma 1. If all answer are negative we have $230 + 20 = 250$ elements and 15 answer. But $N_{15}(2) \geq 252$.

$\square$

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