Fourteenth International Workshop on Algebraic and Combinatorial Coding Theory
September 7–13, 2014, Svetlogorsk (Kaliningrad region), Russia *pp. 109–114*

On the Riesz energy of spherical designs

P. Boyvalenkov[1] peter@math.bas.bg
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
P. Dragnev[2] dragnevp@ipfw.edu
Department of Mathematical Sciences, IPFW, Fort Wayne, IN, USA
D. Hardin, E. Saff[3] {edward.b.saff,doug.hardin}@vanderbilt.edu
Department of Mathematics, Vanderbilt University, Nashville, TN, USA
M. Stoyanova[4] stoyanova@fmi.uni-sofia.bg
Faculty of Mathematics and Informatics, Sofia University, Bulgaria

Abstract. We show how polynomial techniques can be applied for obtaining upper and lower bounds on the Riesz energy of spherical designs.

1 Introduction

A spherical τ-design $C \subset \mathbb{S}^{n-1}$ is a finite nonempty subset of $\mathbb{S}^{n-1}$ such that

$$\frac{1}{\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} f(x)d\mu(x) = \frac{1}{|C|}\sum_{x\in C} f(x) \tag{1}$$

($\mu(x)$ is the Lebesgue measure) holds for all polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree at most τ. The number $\tau = \tau(C)$ is called strength of C.

The spherical designs were introduced in 1977 by Delsarte-Goethals-Seidel [5] where the authors proved that the minimum possible cardinality of a τ-design on $\mathbb{S}^{n-1}$ is at least

$$D(n,\tau) = \begin{cases} 2\binom{n+e-2}{n-1}, & \text{if } \tau = 2e-1, \\ \binom{n+e-1}{n-1} + \binom{n+e-2}{n-1}, & \text{if } \tau = 2e. \end{cases}$$

[1]This author is also with South-Western University (Faculty of Mathematics and Natural Sciences), Blagoevgrad, Bulgaria. His research was supported by the Bulgarian NSF under Contract I01/0003.

[2]The research of this author was supported, in part, by the Simons Foundation CGM no. 282207.

[3]The research of these authors was supported, in part, by the U. S. National Science Foundation under grant DMS-1109266.

[4]The research of this author was supported by the SF of Sofia University under Contract 015/2014.

Let $\alpha > 0$, $C \subset \mathbb{S}^{n-1}$ be a spherical τ-design and

$$W(C,n,\alpha) = \sum_{x,y \in C, x \neq y} [d(x,y)]^{-2\alpha} = \sum_{x,y \in C, x \neq y} [2(1-\langle x,y \rangle)]^{-\alpha}$$

be the Riesz energy of C. Denote by

$$W(N,n,\tau,\alpha) = \inf\{W(C,n,\alpha) : |C| = N, C \subset \mathbb{S}^{n-1}, C \text{ is } \tau\text{-design}\}$$

the minimum possible α-energy of spherical τ-designs on $\mathbb{S}^{n-1}$ of N points. Denote also $h(t) = [2(1-t)]^{-\alpha}$.

Configurations with minimal or near minimal Riesz energy have been a source of many investigations (see [1], [4], [6], [11]). In this regard estimations on the quantity $W(N,n,\tau,\alpha)$ are important. Energy of designs on $\mathbb{S}^2$ were considered in [7, 8].

In this note we show how polynomial techniques can be applied for obtaining lower and upper bounds on $W(N,n,\tau,\alpha)$ and give some examples for small τ.

2 Some preliminaries

For fixed dimension n, the Gegenbauer polynomials [12] are defined by $P_0^{(n)} = 1$, $P_1^{(n)} = t$ and the three-term recurrence relation

$$(k+n-2)P_{k+1}^{(n)}(t) = (2k+n-2)tP_k^{(n)}(t) - kP_{k-1}^{(n)}(t) \text{ for } k \geq 1.$$

We note that $\{P_i^{(n)}(t)\}$ are orthogonal in $[-1,1]$ with a weight $(1-t^2)^{(n-3)/2}$. If $f(t) \in \mathbb{R}[t]$ is a real polynomial of degree k then $f(t)$ can be uniquely expanded in terms of the Gegenbauer polynomials as $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$. The coefficients f_i, $i = 0,1,\ldots,k$, are important in the so-called linear programming theorems. The identity

$$|C|f(1) + \sum_{x,y \in C, x \neq y} f(\langle x,y \rangle) = |C|^2 f_0 + \sum_{i=1}^{k} \frac{f_i}{r_i} \sum_{j=1}^{r_i} \left(\sum_{x \in C} v_{ij}(x) \right)^2 \qquad (2)$$

is an important source of estimations by polynomial techniques. Here $C \subset \mathbf{S}^{n-1}$ is a spherical code, $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ as above, $\{v_{ij}(x) : j = 1,2,\ldots,r_i\}$ is an orthonormal basis of the space $\mathrm{Harm}(i)$ of homogeneous harmonic polynomials of degree i and $r_i = \dim \mathrm{Harm}(i)$. In the classical case (cf. [5, 9]) the sums of the both sides are neglected for suitable polynomials.

The identity (2) can be used for estimations of the Riesz energy of spherical codes (see [1]). Here we show that either lower and upper bound are possible when the code is a τ-design with positive τ.

3 General bounds

The general frame of the linear programming bounds on $W(N,n,\tau,\alpha)$ is given by the next two theorems.

An equivalent definition of spherical designs says that $\sum_{x\in C} v_{ij}(x)=0$ for every $i\le\tau$ and every $j\le r_i$ (cf. [9]). This suggests that polynomials of degree at most τ could be useful – the right hand side of (2) is then reduced to $|C|^2 f_0$.

Theorem 1. *Let N, n, α and τ be fixed and $f(t)$ be a real polynomial such that*

(A0) $\deg(f)\le\tau$*;*

(A1) $f(t)\le h(t)$ *for* $-1\le t\le 1$*.*

Then $W(N,n,\tau,\alpha)\ge N(f_0N-f(1))$*.*

Proof. Let $C\subset\mathbb{S}^{n-1}$ be an arbitrary spherical τ-design of $|C|=N$ points. We consecutively have

$$\begin{aligned} Nf(1)+W(C,n,\alpha) &= Nf(1)+\sum_{x,y\in C,x\neq y} h(\langle x,y\rangle)\ge |C|f(1)+\sum_{x,y\in C,x\neq y} f(\langle x,y\rangle)\\ &= |C|^2 f_0+\sum_{i=1}^{k}\frac{f_i}{r_i}\sum_{j=1}^{r_i}\left(\sum_{x\in C} v_{ij}(x)\right)^2 = N^2 f_0, \end{aligned}$$

which implies that $W(C,n,\alpha)\ge N(f_0N-f(1))$. Since the design C was arbitrary, we conclude that $W(N,n,\tau,\alpha)\ge N(f_0N-f(1))$. □

Theorem 2. *Let N, n, α and τ be fixed and $g(t)$ be a real polynomial such that*

(A0) $\deg(g)\le\tau$*;*

(A1') $g(t)\ge h(t)$ for $-1\le t\le t_0$, and $g(t)\le h(t)$ for $t\in[t_0,1)$ where t_0 is such that no τ-design on $\mathbb{S}^{n-1}$ of N points can have inner products in the interval $(t_0,1)$.

Then $W(N,n,\tau,\alpha)\le N(g_0N-g(1))$*.*

Proof. Let $C\subset\mathbb{S}^{n-1}$ be an arbitrary spherical τ-design of $|C|=N$ points. We consecutively have

$$\begin{aligned} Ng(1)+W(C,n,\alpha) &= Ng(1)+\sum_{x,y\in C,x\neq y} h(\langle x,y\rangle)\le |C|g(1)+\sum_{x,y\in C,x\neq y} g(\langle x,y\rangle)\\ &= |C|^2 g_0+\sum_{i=1}^{k}\frac{g_i}{r_i}\sum_{j=1}^{r_i}\left(\sum_{x\in C} v_{ij}(x)\right)^2 = N^2 g_0, \end{aligned}$$

which implies that $W(C,n,\alpha)\le N(g_0N-g(1))$. Since the design C was arbitrary, we conclude that $W(N,n,\tau,\alpha)\le N(g_0N-g(1))$. □

We denote by $A_{n,\tau,\alpha}$ ($B_{n,\tau,\alpha}$ resp.) the set of suitable polynomials for Theorem 1 (Theorem 2 resp.). Then Theorems 1 and 2 imply that

$$\sup_{f\in A_{n,\tau,\alpha}} N(f_0N - f(1)) \leq W(N,n,\tau,\alpha) \leq \inf_{g\in B_{n,\tau,\alpha}} N(g_0N - g(1)).$$

We show examples with lower and upper bounds for the Riesz energy of 2-designs below.

4 Applications

The next lemma is useful in dealing with the conditions (A1) and (A1').

Lemma 1. *The equation $f(t) = h(t)$ can not have more than $1 + \deg(f)$ roots (counted with multiplicities).*

Proof. Let $\deg(f) = k$. The $(k+1)$-th derivative of the function $h(t) - f(t)$ is $h^{(k+1)}(t) = \frac{2^{k+1}\alpha(\alpha+1)\ldots(\alpha+k)}{[2(1-t)]^{\alpha+k+1}} > 0$. Then by Rolle's theorem the k-th derivative $h^{(k)}(t) - f^{(k)}(t)$ can have at most one zero and so on, finally the function $h(t) - f(t)$ can have at most $k+1$ zeros. □

4.1 Lower bounds for 2-designs

We look for a polynomial $f(t) = a_0 + a_1t + a_2t^2$ such that $f(-1) = h(-1)$, $f(b) = h(b)$ and $f'(b) = h'(b)$ for some $b \in [-1,1]$. It follows from Lemma 1 (together with $a_2 > 0$) that $h(t) \geq f(t)$ for every $t \in [-1,1)$ with equality iff $t = -1$ or $t = b$.

Further, we have

$$f(t) = h(b) + h'(b)(t-b) + \frac{[h'(b)(1+b) - h(b) + h(-1)](t-b)^2}{(1+b)^2}.$$

We continue with the function $\Phi_2(b) = Nf_0 - f(1) = \frac{A}{n(1+b)^2(1-b)}$, where

$$\begin{aligned} A \;=\; & N[\alpha h(b)(1+b)(1-nb) - h(b)(1-b)(1-n-2nb) \\ & + h(-1)(1-b)(1+nb^2)] \\ & - n(1-b)[2\alpha h(b)(1+b) + 4bh(b) + h(-1)(1-b)^2]. \end{aligned}$$

The equation $\Phi_2'(b) = 0$ gives stationary point $b_2 = -\frac{2n-N}{n(N-2)}$ which appears to set the maximum of $\Phi_2(b)$. We now calculate

$$W(N,n,\alpha,2) \geq N\Phi_2(b_2) = \frac{2^\alpha n^{\alpha+1}(N-2)^{\alpha+2} + N^{\alpha+1}(N-n-1)(n-1)^\alpha}{4^\alpha N^{\alpha-1}(n-1)^\alpha[N(n+1)-4n]} \tag{3}$$

for every $N \geq n+1$, $n \geq 3$ and $\alpha > 0$. This bound is attained by the so-called bi-orthogonal code which is in fact a 3-design. In this case we have

$$W(2n,n,\alpha,2) = W(2n,n,\alpha,3) = N\Phi(b_2) = \frac{2n[(n-1)2^{\alpha+1}+1]}{4^\alpha}.$$

4.2 Upper bounds for 2-designs

Suitable t_0 for Theorem 2 are the upper bounds on

$$s(N,\tau) = \max\{s(C) : C \subset \mathbb{S}^{n-1} \text{ is a } \tau\text{-design}, |C| = N\},$$

where $s(C) = \max\{\langle x,y\rangle : x,y \in C, x \neq y\}$. We show such bounds for $\tau = 2$.

Lemma 2. [3] *We have*

$$s(N,2) \leq \frac{N-2}{n} - 1.$$

Proof. This bound follows from Theorem 3.2 from [3] with suitable polynomial of second degree. □

For even τ and cardinality $N < D(n,\tau+1)$ we have also lower bounds on

$$\ell(N,\tau) = \min\{\ell(C) : C \subset \mathbb{S}^{n-1} \text{ is a } \tau\text{-design}, |C| = N\},$$

where $\ell(C) = \min\{\langle x,y\rangle : x,y \in C, x \neq y\}$.

Lemma 3. [3] *We have*

$$\ell(N,2) \geq 1 - \frac{N}{n}.$$

Proof. This bound follows from Theorem 3.3 from [3] with suitable polynomial of second degree. □

Other bounds on $s(N,\tau)$ and $\ell(N,\tau)$ can be obtained as in Section 4 of [2] with suitable polynomials. For $\tau = 2$ such bounds are worse than these from Lemmas 2 and 3 but become better for larger even τ.

We denote by s and ℓ the minimum and the maximum in the right hand side of the estimations of Lemmas 2 and 3, respectively. Then a linear polynomials which graph passes through the points $(\ell, h(\ell))$ and $(s, h(s))$ satisfies the conditions of Theorem 2 and gives the upper bound

$$W(N,n,\alpha,2) \leq \frac{N[[s(1-s)^\alpha - \ell(1-\ell)^\alpha](N-1) - (1-\ell)^\alpha + (1-s)^\alpha]}{2^\alpha(1-s)^\alpha(1-\ell)^\alpha(s-\ell)}.$$

This bound can be used, for example, for proving the nonexistence of 2-designs of $n+2$ points on $\mathbb{S}^{n-1}$ for odd n (see [10]).

References

[1] S. Borodachov, D. Hardin, E. Saff, *Minimal Discrete Energy on the Sphere and other Manifolds*, manuscript.

[2] P. Boyvalenkov, S. Bumova, D. Danev, Necessary conditions for existence of some designs in polynomial metric spaces, *Europ. J. Combin.*, **20** 213-225, 1999.

[3] S. Boumova, P. Boyvalenkov, H. Kulina, M. Stoyanova, Polynomial techniques for investigation of spherical designs, *Des., Codes Crypt.*, **51** 275-288, 2009.

[4] H. Cohn, A. Kumar, Universally optimal distribution of points on spheres, *J. of AMS*, **20** no. 1, 99-148, 2006.

[5] P. Delsarte, J.-M. Goethals, J. J. Seidel, Spherical codes and designs, *Geom. Dedicata,* **6** 363-388, 1977.

[6] D. P. Hardin, E. B. Saff, Discretizing manifolds via minimum energy points, *Notices AMS*, **51**, no. 10, 1186-1194, 2004.

[7] K. Hesse, The s-energy of spherical designs on $\mathbb{S}^2$, *Adv. Comput. Math.*, **30**, 37-59, 2009.

[8] K. Hesse, P. Leopardi, The Coulomb energy of spherical designs on $\mathbb{S}^2$, *Adv. Comput. Math.*, **28**, 331-354, 2008.

[9] V. I. Levenshtein, Universal bounds for codes and designs, *Handbook of Coding Theory*, V. S. Pless and W. C. Huffman, Eds., Elsevier, Amsterdam, 1998, Ch. 6, 499–648.

[10] Y. Mimura, A construction of spherical 2-designs, *Graphs Combin.*, **6** 369-372, 1990.

[11] E. B. Saff, A. B. J. Kuijlaars, Distributing many points on a sphere, *Math. Intelligencer,* bf 19 5-11, 1997.

[12] G. Szegö, *Orthogonal polynomials*, AMS Col. Publ., **23**, Providence, RI, 1939.