Fourteenth International Workshop on Algebraic and Combinatorial Coding Theory
September 7–13, 2014, Svetlogorsk (Kaliningrad region), Russia *pp. 80–84*

Homogeneous Arcs in Projective Hjelmslev Spaces [1]

Stoyan Boev stoyan@nbu.bg
New Bulgarian University, 21 Montevideo str., 1618 Sofia, Bulgaria
Ivan Landjev i.landjev@nbu.bg
New Bulgarian University, 21 Montevideo str., 1618 Sofia, Bulgaria

Abstract. In this note, we define homogeneous arcs in projective Hjelmlsev spaces. They are multisets of points with respect to the homogeneous weight introduced by W. Heise and I. Constantinescu. These arcs turn out to be equivalent with the linear codes of full length over finite chain rings with respect to the homogeneous weight. We formulate and prove an analogue of the classical theorem by A. Bonisoli which characterizes the arcs with constant intersection number. In this case, the constant (homogeneous) weight arcs are sums of neighbour classes of points.

1 Introduction

Let R be a finite chain ring with $|R| = q^2$, $R/\operatorname{rad} R \cong \mathbb{F}_q$. Consider the k-dimensional projective Hjelmslev geometry $\Pi_k = \mathrm{PHG}({}_RR^{k+1})$. A multiset $\mathcal{K}$ in Π_k is a mapping $\mathcal{K} : \mathcal{P} \to \mathbb{N}_0$, where $\mathcal{P}$ denotes the pointset of Π_k. A multiset $\mathcal{K}$ is called an (N, w)-arc if $\mathcal{K}(\mathcal{P}) = N$, $\mathcal{K}(H) \leq w$, for every hyperplane H, and there exists a hyperplane H_0 with $\mathcal{K}(H_0) = w$. It is well-known that there is a one-to-one correspondence between the isomorphism classes of linear codes of full length over R and the classes of projectively equivalent multisets in the geometries Π_k. For the structure and general properties of the projective Hjelmslev geometries we refer to [4, 5].

During the years it turned out that the Hamming metric is not good for the study of ring codes. The minimum Hamming distance of a code is equal to the Hamming distance of a rather small subcode – the radical of the code. Thus it is more appropriate to consider linear codes over rings with respect to the homogeneous weight (distance) introduced by Constantinescu and Heise in [2].

In this note we introduce a new weight for arcs in projective Hjelmslev geometries over chain rings of nilpotency index 2, and prove an analogue to a well-known result by A. Bonisoly on arcs with a constant multiplicity of all hyperplanes.

[1] The research of the second author has been supported by the Scientific Research Fund of Sofia University under Contract 37/28.04.2014.

2 Homogeneous Arcs

We start by introducing a homogeneous weight for arcs in Π_k. For every subspace S in Π_k, set

$$\omega(S) = \mathcal{K}(S) - \frac{1}{q-1}\mathcal{K}([S] \setminus S). \tag{1}$$

A multiset $\mathcal{K}$ is called a homogeneous (N, W)-arc if $\mathcal{K}(\mathcal{P}) = N$, $\omega(H) \leq W$ for every hyperplane H, and there exists a hyperplane H_0 with $\omega(H_0) = W$. Similarly, a homogeneous (N, W)-blocking set is a multiset $\mathcal{K}$ with $\mathcal{K}(\mathcal{P}) = N$, $\omega(H) \leq W$ for every hyperplane H, and $\omega(H_0) = W$ for some hyperplane H_0. An arc (blocking set) is called projective if the multiplicity of every point is 0 or 1. As in the classical case, the complement of a projective arc is a projective blocking set.

Theorem 1. *Let $\mathcal{K}$ be a projective arc with homogeneous weights $W_1 < W_2 < \ldots < W_s$. Then the complementary arc $\Pi_k \backslash \mathcal{K}$ has weights $-W_s < \ldots < -W_2 < -W_1$.*

Example 1. *In [3] a class of $(q^2+q+1, 2)$-arcs are constructed for the planes over the Galois rings of nilpotency index 2 and characteristic 4. These are homogeneous two-weight arcs with $W_1 = -\frac{q+1}{q-1}$, $W_2 = 1$. The complement has weight $W_1' = -1$, $W_2' = \frac{q+1}{q-1}$.*

Let us recall that a code over $\mathbb{F}_q$ (not necessarily linear) is called linearly representable if it can be obtained as the q-ary image of an R-linear code under the Reed-Solomon map:

$$\psi_{\mathrm{RS}}\colon \left\{ \begin{array}{ccc} R & \to & \mathbb{F}_q^q, \\ r = r_0 + r_1\theta & \to & (r_0, r_1)A, \end{array} \right.$$

where

$$A = \begin{pmatrix} 0 & 1 & \zeta & \ldots & \zeta^{q-2} \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}$$

Here θ is a generator of $\operatorname{rad} R$, ζ is a primitive element of $\mathbb{F}_q$ and r_i are chosen from and arbitrarily fixed set Γ of q elements from R no two of which are congruent modulo $\operatorname{rad} R$. The importance of the homogeneous weight for arcs is due to the following fact.

Theorem 2. *A linearly representable q-ary code with parameters $(Nq, q^{2k}, (q-1)(N-W))$ exists if and only if there exists a homogeneous (N, W)-arc in Π_k, whose support generates the whole geometry Π_k.*

3 A Generalization of a Theorem by Bonisoli

In [1] A. Bonisoli proved that a linear code in which all non-zero words assume the same weight is a direct sum of simplex codes. For arcs in $\mathrm{PG}(k-1,q)$ such that all hyperplanes have the same multiplicity are the sum of several copies of the whole projective space. In the next theorem, we prove an analogue of Bonisoli's theorem for the homogeneous weight introduced in (1).

Lemma 1. *Let $\mathcal{K}$ be an (N,W)-homogeneous arc in Π_k. Then*

$$\sum_H \omega(H) = 0,$$

where the sum is over all hyperplanes of Π_k.

The proof is obtained by simple counting. This lemma implies that if the homogeneous weight of all hyperplanes is the same then it must be 0.

Theorem 3. *Every $(N,0)$-homogeneous arc is a sum of neighbour classes of points.*

Proof. Order linearly the points x_i and the hyperplanes H_i, $i = 0,1,\ldots,q^{k-1}\frac{q^k-1}{q-1} - 1$, in such way that $x_i \frown x_j$ (resp. $H_i \frown H_j$) iff $\lfloor i/q \rfloor = \lfloor j/q \rfloor$. For this linear ordering of points and hyperplanes, define the square matrix $A = (a_{ij})$ of size $q^{k-1}\frac{q^k-1}{q-1}$ by

$$a_{ij} = \begin{cases} 1 & \text{if } x_i \in H_j, \\ -\frac{1}{q-1} & \text{if } x_i \notin H_j \text{ but } x_i \frown H_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

With every homogeneous arc $\mathcal{K}$ we associate a vector

$$\mathbf{x}_{\mathcal{K}} = \left(\mathcal{K}(x_0), \mathcal{K}(x_1) \ldots, \mathcal{K}(x_{q^{k-1}\frac{q^k-1}{q-1}-1})\right).$$

If $\mathcal{K}$ is a $(N,0)$ arc then $\mathbf{x}_{\mathcal{K}} A = \mathbf{0}$, where $\mathbf{0} = (0,0,\ldots,0)$ is of length $q^{k-1}\frac{q^k-1}{q-1}$.

Now we are going to prove that $\operatorname{rk} A = (q^{k-1}-1)\frac{q^k-1}{q-1}$. Consider the matrix $A' = (a|B)$, where B is a $q^{k-1}\frac{q^k-1}{q-1}$ by $\frac{q^k-1}{q-1}$ matrix whose columns are the incidence vectors of the neighbour classes of points:

$$b_{ij} = \begin{cases} 1 & \text{if } x_i \in [x_{jq^{k-1}}] \\ 0 & \text{if } x_i \notin [x_{jq^{k-1}}]. \end{cases}$$

The characteristic vectors $\chi(H_i)$ of the hyperplanes of Π_k belong to the vector space spanned by the columns of A'. Since the incidence matrix of all

s-dimensional versus all t-dimensional Hjelmslev subspaces is of full rank, the matrix A' is of full rank [6], i.e.

$$\operatorname{rk} A' = q^{k-1}\frac{q^k-1}{q-1}.$$

This implies that

$$\operatorname{rk} A \geq q^{k-1}\frac{q^k-1}{q-1} - \frac{q^k-1}{q-1} = (q^{k-1}-1)\frac{q^k-1}{q-1}.$$

On the other hand, we have

$$\sum_{j:\lfloor j/q^{k-1}=a\rfloor} A^{(j)} = \mathbf{0},$$

for all $a \in \{0, 1, \dots, \frac{q^k-1}{q-1} - 1\}$. Here $A^{(j)}$ are the columns of A. This implies that

$$\operatorname{rk} A = (q^{k-1}-1)\frac{q^k-1}{q-1},$$

and the space of all solutions is spanned by the vectors

$$\mathbf{b}_a = (\underbrace{0, 0, \dots, 0}_{aq^{k-1}}, \underbrace{1, 1, \dots, 1}_{q^{k-1}}, \underbrace{0, 0, \dots, 0}_{q^{k-1}\frac{q^k-1}{q-1}-q-1}),$$

where $a = 0, 1, \dots, \frac{q^k-1}{q-1} - 1$. □

References

[1] A. Bonisoli, Ecery equidistant linear code is a sequence of dual Hamming codes, *Ars Combinatoria* **18**(1984), 181-186.

[2] I. Constantinescu, W. Heise, A metric for codes over residue class rings, *Problemns of Information Transmission* **33**(3)(1997), 208–213.

[3] Th. Honold, I. Landjev, On maximal arcs in projective Hjelmslev planes over chain rings of even characteristic, *Finite Fields and Their Applications* **11**(2005), 292–304.

[4] Th. Honold, I. Landjev, Codes over rings and ring geometries, chapter 7 in: "Current research topics in Galois geometries" (eds. L.Storme and Jan De Beule) NOVA Publishers, 2012, 161-186. (ISBN 978-1-61209-523-3)

[5] Th. Honold, I. Landjev, Codes over Rings and Ring Geometries, in: Current research Topics in Galois Geometries (L. Storme, J De Beule eds.), NOVA Publishers, New York, 2012.

[6] I. Landjev, P. Vanderdriesche, On the rank of incidence matrices in projective Hjelmslev spaces, Designs, Codes and Cryptography, 2014, to appear.