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Linear Codes associated to Determinantal Varieties [1]

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Abstract. We consider a class of linear codes associated to projective algebraic varieties defined by the vanishing of minors of a fixed size of a generic matrix. It is seen that the resulting code has only a small number of distinct weights. The case of varieties defined by the vanishing of 2×2 minors is considered in some detail. Here we obtain the complete weight distribution. Moreover, several generalized Hamming weights are determined explicitly and it is shown that the first few of them coincide with the distinct nonzero weights.

1 Introduction

A useful and interesting way to construct a linear code is to consider a projective algebraic variety V defined over the finite field $\mathbb{F}_q$ with q elements together with a nondegenerate embedding in a projective space, and to look at the projective system (in the sense of Tsfasman and Vlăduţ [10]) associated to the $\mathbb{F}_q$-rational points of V. A good illustration is provided by the case of Grassmann codes and Schubert codes, which have been of much interest; see, for example, [6], [3], [4], [13] or the survey [8]. In this paper we consider a class of linear codes that are associated to classical determinantal varieties. These will be referred to as determinantal codes. The length and dimension of these

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codes are easy to determine and also one can readily show that they are nondegenerate. We shall then focus on the question of determining the minimum distance and more generally, the complete weight disribution, and also the generalized Hamming weights of determinantal codes. From a geometric viewpoint, this corresponds to determining the number of $\mathbb{F}_q$-rational points in all possible hyperplane sections and also in maximal linear sections of determinantal varieties. We give a general description of all the weights of determinantal codes and then analyze in greater details the codes associated to the variety defined by the vanishing of all 2×2 minors of a generic $\ell \times m$ matrix. It is seen in this case that the codes exhibit a curious phenomenon that there are exactly ℓ nonzero weights and these coincide with the first ℓ generalized Hamming weights which happen to meet the Griesmer-Wei bound. This phenomenon is exhibited by $[n, k]_q$-MDS code (for instance, the Reed-Solomon codes), which have exactly k nonzero weights and k generalized Hamming weights given by $n - k + 1, \dots, n$. Another trivial example is that of the simplex code (i.e., the dual of Hamming code) which has only one nonzero weight and it evidently coincides the first generalized Hamming weight. However, we do not know other nontrivial examples and determinantal codes appear to be intersting in this regard. Unlike simplex codes, determining *all* generalized Hamming weights of determinantal codes seems difficult. but we make some partial progress here.

It turns out (although we were not initially aware of it) that codes analogous to determinantal codes were considered in a different context by Camion [1] and Delsarte [2] who consider codes derived from bilinear forms. In effect, Delsarte obtains the weight distribution of these codes using an explicit determination of the characters of the Schur ring of an association scheme corresponding to these bilinear forms (see end of §3 below for more details). Our approach, however, is entirely different and may be of some interest. Further, results concerning generalized Hamming weights appear to be new and the auxiliary results used here may be of some independent interest.

This article is an extended abstract and thus the proofs are skipped. But all the definitions, statements of lemmas and theorems are completely given.

2 Preliminaries

Fix throughout this paper a prime power q, positive integers t, ℓ, m, and a $\ell \times m$ matrix $X = (X_{ij})$ whose entries are independent indeterminates over $\mathbb{F}_q$. We will denote by $\mathbb{F}_q[X]$ the polynomial ring in the ℓm variables X_{ij} ($1 \le i \le \ell$, $1 \le j \le m$) with coefficients in $\mathbb{F}_q$. As usual, by a *minor* of size t or a $t \times t$ minor of X we mean the determinant of a $t \times t$ submatrix of X, where t is a nonnegative integer $\leq \min\{\ell, m\}$. As per standard conventions, the only 0×0 minor of X is 1. We will be mostly interested in the class of minors of a fixed size, and this class is unchanged if X is replaced by its transpose. With this in view, we shall always assume, without loss of generality, that $\ell \leq m$. Given a field $\mathbb{F}$, we denote by $\mathbb{M}_{\ell\times m}(\mathbb{F})$ the set of all $\ell \times m$ matrices with entries in $\mathbb{F}$. Often $\mathbb{F} = \mathbb{F}_q$ and in this case we may simply write $\mathbb{M}_{\ell\times m}$ for $\mathbb{M}_{\ell\times m}(\mathbb{F}_q)$. Note that $\mathbb{M}_{\ell\times m}$ can be viewed as an affine space $\mathbb{A}^{\ell m}$ over $\mathbb{F}_q$ of dimension ℓm. For $0 \leq t \leq \ell$, the corresponding classical determinantal variety (over $\mathbb{F}_q$) is denoted by $\mathcal{D}_t$ and defined as the affine algebraic variety in $\mathbb{A}^{\ell m}$ given by the vanishing of all $(t+1)\times(t+1)$ minors of X; in other words

$$\mathcal{D}_t = \{M \in \mathbb{M}_{\ell\times m}(\mathbb{F}_q) : \operatorname{rank}(M) \leq t\}\,.$$

The affine variety $\mathcal{D}_t$ is, in fact, a cone; in other words, the vanishing ideal $\mathcal{I}_{t+1}$ (which is precisely the ideal of $\mathbb{F}_q[X]$ generated by all $(t+1)\times(t+1)$ minors of X) is a homogeneous ideal. Also it is a classical (and nontrivial) fact that $\mathcal{I}_{t+1}$ is a prime ideal. Thus $\mathcal{D}_t$ can also be viewed as a projective algebraic variety in $\mathbb{P}^{\ell m-1}$, and viewed this way, we will denote it by $\widehat{\mathcal{D}}_t$. We remark that the dimension of $\widehat{\mathcal{D}}_t$ as a projective variety is $t(\ell+m-t)-1$. Briefly put, the determinantal code $\widehat{C}_{\mathrm{det}}(t;\ell,m)$ is the linear code corresponding to the projective system $\widehat{\mathcal{D}}_t \hookrightarrow \mathbb{P}^{\ell m-1}(\mathbb{F}_q) = \mathbb{P}(\mathbb{M}_{\ell\times m})$. An essentially equivalent way to obtain this code is to consider the image $C_{\mathrm{det}}(t;\ell,m)$ of the evaluation map

$$\mathrm{Ev}: \mathbb{F}_q[X]_1 \to \mathbb{F}_q^n \quad \text{defined by} \quad \mathrm{Ev}(f) = c_f := (f(M_1),\ldots,f(M_n))\,, \qquad (1)$$

where $\mathbb{F}_q[X]_1$ denotes the space of homogeneous polynomials in $\mathbb{F}_q[X]$ of degree 1 together with the zero polynomial, and $M_1,\ldots,M_n$ is an ordering of $\mathcal{D}_t$.

Recall that in general for a linear code C of length n, i.e., for a linear subspace C of $\mathbb{F}_q^n$, the *Hamming weight* of a codeword $c = (c_1,\ldots,c_n)$, denoted $\mathrm{w_H}(c)$, and the *support weight* of any $D \subseteq C$, denoted $\|D\|$, are defined by

$$\mathrm{w_H}(c) := |\{i : c_i \neq 0\}| \quad \text{and} \quad \|D\| := |\{i : \text{there exists } c \in C \text{ with } c_i \neq 0\}|,$$

where for a finite set S, by $|S|$ we denote the cardinality of S. The minimum distance of C, denoted $d(C)$, and more generally, the r^{th} *higher weight* or the r^{th} *generalized Hamming weight* of C, denoted $d_r(C)$, are defined by

$$\begin{aligned} d(C) &:= \min\{\mathrm{w_H}(c) : c \in C,\ c \neq 0\} \quad \text{and for } r = 1,\ldots,k, \\ d_r(C) &:= \min\{\|D\| : D \text{ is a subcode of } C \text{ with } \dim D = r\}. \end{aligned}$$

The parameters of $C_{\mathrm{det}}(t;\ell,m)$ determine those of $\widehat{C}_{\mathrm{det}}(t;\ell,m)$ and vice-versa. More precisely, we have the following.

Proposition 1. *Write $C = C_{\det}(t;\ell,m)$ and $\widehat{C} = \widehat{C}_{\det}(t;\ell,m)$. Let n, k, d, and A_i (resp. $\hat{n}$, $\hat{k}$, $\hat{d}$, and $\hat{A}_i$) denote, respectively, the length, dimension, minimum distance and the number of codewords of weight i of C (resp. $\widehat{C}$). Then*

$$n = 1 + \hat{n}(q-1), \quad k = \hat{k}, \quad d = \hat{d}(q-1), \quad \text{and} \quad A_{i(q-1)} = \hat{A}_i \text{ for } 0 \le i \le \hat{n}.$$

Moreover $A_n = 0$ and more generally, $A_j = 0$ for $0 \le j \le n$ such that $(q-1) \nmid j$. Furthermore, if for $1 \le r \le k$, we denote by d_r and $A_i^{(r)}$ (resp: $\hat{d}_r$ and $\hat{A}_i^{(r)}$) the r^{th} higher weight *and the number of r-dimensional subcodes of support weight i of C (resp. $\widehat{C}$), then $d_r = (q-1)\hat{d}_r$ and $A_{i(q-1)}^{(r)} = \hat{A}_i^{(r)}$ for $0 \le i \le \hat{n}$.*

The code $C_{\det}(t;\ell,m)$ is degenerate, whereas $\widehat{C}_{\det}(t;\ell,m)$ is nondegenerate. The length and dimension of these two codes are easily obtained. The former goes back at least to Landsberg (1893) who obtained a formula for n, or rather the number of matrices in $\mathbb{M}_{\ell\times m}$ of a given rank t in case q is prime.

Proposition 2. *$\widehat{C}_{\det}(t;\ell,m)$ is nondegenerate of dimension $\hat{k} = \ell m$ and length*

$$\hat{n} = \sum_{j=1}^{t} \hat{\mu}_j(\ell,m) \quad \text{where} \quad \hat{\mu}_j(\ell,m) := \frac{q^{\binom{j}{2}}}{q-1} \prod_{i=0}^{j-1} \frac{\left(q^{\ell-i}-1\right)\left(q^{m-i}-1\right)}{q^{i+1}-1}.$$

Determining the minimum distance of $\widehat{C}_{\det}(t;\ell,m)$ isn't quite obvious. To get some feel for this, let us work out some simple examples and also observe that a bound can be readily obtained in a special case.

Example 1. (i) If $\ell = 1$ or if $\ell = m = t$, then $\widehat{\mathcal{D}}_t = \mathbb{P}^{\ell m-1}$ and $\widehat{C}_{\det}(t;\ell,m)$ is a first order projective Reed-Muller code (cf. [8]), and in fact, a simplex code. Evidently, it has length $(q^{\ell m}-1)/(q-1)$ and minimum distance $q^{\ell m-1}$.

(ii) If $\ell = m = t+1$, then $\mathcal{D}_t = \mathbb{M}_{\ell\times m} \setminus \mathrm{GL}_\ell(\mathbb{F}_q)$ while $\widehat{\mathcal{D}}_t$ is the hypersurface in $\mathbb{P}^{\ell^2-1}$ given by $\det(X) = 0$. We can use Serre's inequality (cf. [9]) to obtain the following bound for the minimum distance, say $\hat{d}$, of the code $\widehat{C}_{\det}(t;\ell,\ell)$:

$$\hat{d} \ge q^{\ell^2-1} + q^{\ell^2-2} - (\ell-1)q^{\ell^2-3} - q^{\binom{\ell}{2}} \prod_{i=2}^{\ell} (q^i - 1).$$

In the special case when $\ell = m = 2$ and $t = 1$, we find $\hat{d} \ge q^2$ and it is easy to show that this bound is attained, i.e., $d\big(\widehat{C}_{\det}(1;2,2)\big) = q^2$.

3 Weight Distribution

It turns out that the Hamming weights of codewords of $C_{\mathrm{det}}(t;\ell,m)$ as well as $\widehat{C}_{\mathrm{det}}(t;\ell,m)$ are few in number.

Lemma 1. *Let $f(X)=\sum_{i=1}^{\ell}\sum_{j=1}^{m} f_{ij}X_{ij}$ be a linear homogeneous polynomial in $\mathbb{F}_q[X]$. Denote by $F=(f_{ij})$ the coefficient matrix of f. Then the Hamming weights of the corresponding codewords c_f of $C_{\mathrm{det}}(t;\ell,m)$ and $\hat{c}_f$ of $\widehat{C}_{\mathrm{det}}(t;\ell,m)$ depend only on* $\mathrm{rank}(F)$. *In fact, if $r=\mathrm{rank}(F)$, then $\mathrm{w_H}(c_f)=\mathrm{w_H}(c_{\tau_r})$ and $\mathrm{w_H}(\hat{c}_f)=\mathrm{w_H}(\hat{c}_{\tau_r})$, where $\tau_r := X_{11}+\cdots+X_{rr}$ is the r^{th} partial trace of X.*

Corollary 1. *Each of the codes $C_{\mathrm{det}}(t;\ell,m)$ and $\widehat{C}_{\mathrm{det}}(t;\ell,m)$ have at most $\ell+1$ distinct weights, $w_0,w_1,\dots,w_\ell$ and $\hat{w}_0,\hat{w}_1,\dots,\hat{w}_\ell$ respectively, given by $w_r=\mathrm{w_H}(c_{\tau_r})$ and $\hat{w}_r=\mathrm{w_H}(\hat{c}_{\tau_r})=w_r/(q-1)$ for $r=0,1,\dots,\ell$. Moreover, the weight enumerator polynomials $A(Z)$ of $C_{\mathrm{det}}(t;\ell,m)$ and $\hat{A}(Z)$ of $\widehat{C}_{\mathrm{det}}(t;\ell,m)$ are given by*

$$A(Z)=\sum_{r=0}^{\ell}\mu_r(\ell,m)Z^{w_r} \quad \textit{and} \quad \hat{A}(Z)=\sum_{r=0}^{\ell}\mu_r(\ell,m)Z^{\hat{w}_r},$$

where $\mu_r(\ell,m)$ is the number of $\ell\times m$ matrices over $\mathbb{F}_q$ of rank r, given by

$$q^{\binom{r}{2}}\prod_{i=0}^{r-1}\frac{\left(q^{\ell-i}-1\right)\left(q^{m-i}-1\right)}{q^{i+1}-1}=\begin{bmatrix} m\\ r\end{bmatrix}_q\prod_{i=0}^{r-1}(q^{\ell}-q^{i})=\begin{bmatrix} \ell\\ r\end{bmatrix}_q\prod_{i=0}^{r-1}(q^{m}-q^{i}).$$

We remark that it is not clear, *a priori*, that the weights w_r are distinct for distinct values of r. Also it isn't clear which of the nonzero weights $w_1,\dots,w_r$ is the least. But the weight distribution or the spectrum is completely determined once we solve the combinatorial problem of counting the number of $\ell\times m$ matrices M over $\mathbb{F}_q$ of rank $\le t$ for which $\tau_r(M)\neq 0$. As indicated in the Introduction, Delsarte [2] solved an essentially equivalent problem of determining the number $N_t(r)$ of $M\in\mathbb{M}_{\ell\times m}(\mathbb{F}_q)$ of rank t with $\tau_r(M)\neq 0$, and showed:

$$N_t(r)=\frac{(q-1)}{q}\left(\mu_t(\ell,m)-\sum_{i=0}^{\ell}(-1)^{t-i}q^{im+\binom{t-i}{2}}\begin{bmatrix} m-i\\ m-t\end{bmatrix}_q\begin{bmatrix} m-r\\ i\end{bmatrix}_q\right),$$

where $\mu_t(\ell,m)$ is as in Corollary 1 above. Consequently, the nonzero weights of $C_{\mathrm{det}}(t;\ell,m)$ are given by $w_r=\sum_{s=1}^{t}N_s(r)$ for $r=1,\dots,\ell$. However, for a fixed t (even in the simple case $t=1$), it is not entirely obvious how $w_1,\dots,w_\ell$ are ordered and which among them is the least. In the next section, we circumvent these difficulties and use a direct approach in the case $t=1$.

4 Case of 2×2 minors

In this section we consider the determinantal variety $\mathcal{D}_1$ defined by the vanishing of all 2×2 minors of X, and show that the weight distribution of the corresponding code is explicitly determined in this case. We begin by recalling an elementary and well-known characterization of rank 1 matrices as outer (or dyadic) products of nonzero vectors.

Proposition 3. *Let $\mathbb{F}$ be a field and let $M \in \mathbb{M}_{\ell\times m}(\mathbb{F})$. Then $\mathrm{rank}(M) = 1$ if and only if there are nonzero (row) vectors $\mathbf{u} \in \mathbb{F}^\ell$ and $\mathbf{v} \in \mathbb{F}^m$ such that $M = \mathbf{u}^T\mathbf{v}$. Moreover, if $\mathbf{u}^T\mathbf{v} = \mathbf{a}^T\mathbf{b}$ for nonzero $\mathbf{u}, \mathbf{a} \in \mathbb{F}^\ell$ and $\mathbf{v}, \mathbf{b} \in \mathbb{F}^m$, then $\mathbf{a} = \lambda\mathbf{u}$ and $\mathbf{b} = \lambda^{-1}\mathbf{v}$ for a unique $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.*

The complete weight distribution of determinantal codes in the case $t = 1$ is given by the following theorem together with Corollary 1. In the statement of the theorem, we restrict to $\widehat{C}_{\mathrm{det}}(t; \ell, m)$, but the corresponding result for $C_{\mathrm{det}}(t; \ell, m)$ when $t = 1$ is readily obtained since $w_r = (q-1)\hat{w}_r$.

Theorem 1. *The nonzero weights of $\widehat{C}_{\mathrm{det}}(1; \ell, m)$ are $\hat{w}_1, \dots, \hat{w}_\ell$, given by*

$$\hat{w}_r = \mathrm{w_H}(\hat{c}_{\tau_r}) = q^{\ell+m-2} + q^{\ell+m-3} + \cdots + q^{\ell+m-r-1} = q^{\ell+m-r-1}\frac{(q^r-1)}{q-1}$$

for $r = 1, \dots, \ell$. In particular, $\hat{w}_1 < \hat{w}_2 < \cdots < \hat{w}_\ell$ and the minimum distance of $\widehat{C}_{\mathrm{det}}(1; \ell, m)$ is $q^{\ell+m-2}$.

Remark 1. It may be noted that the exponent $\ell + m - 2$ of q in the minimum distance $\widehat{C}_{\mathrm{det}}(1; \ell, m)$ is precisely the dimension of the determinantal variety $\widehat{\mathcal{D}}_t$ when $t = 1$. We remark also that the relative distance $\delta = d/n$ of $\widehat{C}_{\mathrm{det}}(1; \ell, m)$ is asymptotically equal to 1 as $q \to \infty$. On the other hand, the rate $R = k/n$ is quite small as $q \to \infty$, but it tends to 1 as $q \to 1$.

We now turn to the determination of the higher weights or the generalized Hamming weights of $\widehat{C}_{\mathrm{det}}(1; \ell, m)$. As remarked in the Introduction, the first ℓ higher weights $\hat{d}_1, \dots, \hat{d}_\ell$ coincide with the nonzero weights $\hat{w}_1, \dots, \hat{w}_\ell$ given by Theorem 1. Of course there are many more higher weights, namely, $\hat{d}_1, \dots, \hat{d}_k$, where $k = \ell m$, that are to be determined. It turns out that it is easy to find the first m of them and also to show that these meet the Griesmer-Wei bound.

Theorem 2. *For $r = 1, \dots, m$, the r^{th} higher weight $\hat{d}_r$ of $\widehat{C}_{\mathrm{det}}(1; \ell, m)$ meets the Griesmer-Wei bound and is given by*

$$\hat{d}_r = q^{\ell+m-2} + q^{\ell+m-3} + \cdots + q^{\ell+m-r-1} = q^{\ell+m-r-1}\frac{(q^r-1)}{q-1}.$$

In particular, if $r \leq \ell$ and $\hat{w}_r$ is as in Theorem 1, then $\hat{d}_r = \hat{w}_r$.

For $m < r < \ell + m$, we can obtain lower and upper bounds for $\hat{d}_r$.

Lemma 2. *Assume that $\ell \geq 2$. Then for $s = 1, \ldots, \ell - 1$, the $(m+s)^{\text{th}}$ higher weight $\hat{d}_{m+s}$ of $\widehat{C}_{\det}(1; \ell, m)$ satisfies*

$$q^{\ell-s-1}\frac{(q^{m+s}-1)}{q-1} = \hat{d}_m + q^{\ell-s-1}\frac{(q^s-1)}{q-1} \leq \hat{d}_{m+s} \leq \hat{d}_m + q^{\ell+m-s-2}\frac{(q^s-1)}{q-1},$$

where $\hat{d}_m$ is as in Theorem 2. In particular, $\hat{d}_m + q^{\ell-2} \leq \hat{d}_{m+1} \leq \hat{d}_m + q^{\ell+m-3}$.

It appears interesting to know whether the higher weights subsequent to $\hat{d}_m$ meet the Griesmer-Wei bound. We will show in Theorem 3 below that this is not the case and, in fact, the exact value of $\hat{d}_{m+1}$ is given by the upper bound in the above lemma. The proof is based on the the auxiliary results below that precede the statement of Theorem 3.

Lemma 3. *Let $\mathbb{F}$ be a field and let $\mathbf{u}, \mathbf{a}, \mathbf{x} \in \mathbb{F}^\ell$ and $\mathbf{v}, \mathbf{b}, \mathbf{y} \in \mathbb{F}^m$ be nonzero vectors such that $\mathbf{u}^T\mathbf{v} + \mathbf{a}^T\mathbf{b} = \mathbf{x}^T\mathbf{y}$. Denote by $\langle \mathbf{u}, \mathbf{a}, \mathbf{x} \rangle$ the subspace of $\mathbb{F}^\ell$ spanned by $\mathbf{u}, \mathbf{a}, \mathbf{x}$, and by $\langle \mathbf{v}, \mathbf{b}, \mathbf{y} \rangle$ the subspace of $\mathbb{F}^m$ spanned by $\mathbf{v}, \mathbf{b}, \mathbf{y}$. Then $\langle \mathbf{u}, \mathbf{a}, \mathbf{x} \rangle$ is one-dimensional or $\langle \mathbf{v}, \mathbf{b}, \mathbf{y} \rangle$ is one-dimensional.*

Corollary 2. *Let $\mathbb{F}$ be a field and let $\mathcal{E}$ be a subspace of $\mathbb{M}_{\ell \times m}(\mathbb{F})$ such that* $\text{rank}(M) = 1$ *for all nonzero $M \in \mathcal{E}$. Then*

$$\mathcal{E} = \left\{\mathbf{u}^T\mathbf{v} : \mathbf{v} \in V\right\} \quad \text{for some } \mathbf{u} \in \mathbb{F}^\ell \text{ and a subspace } V \text{ of } \mathbb{F}^m \tag{2}$$

or

$$\mathcal{E} = \left\{\mathbf{u}^T\mathbf{v} : \mathbf{u} \in U\right\} \quad \text{for some } \mathbf{v} \in \mathbb{F}^m \text{ and a subspace } U \text{ of } \mathbb{F}^\ell. \tag{3}$$

In particular, $\dim \mathcal{E} \leq \max\{\ell, m\} = m$.

Lemma 4. *Let $\mathcal{D}$ be an r-dimensional subspace of $\mathbb{M}_{\ell \times m}(\mathbb{F}_q)$ with $r > m$. Then $\mathcal{D}$ contains at most $q^{r-1} + q^2 - q - 1$ matrices of rank 1. Consequently, $\mathcal{D}$ has at least $\left(q^{r-1} - q\right)(q-1)$ matrices of rank ≥ 2.*

Theorem 3. *Assume that $\ell \geq 2$. For $1 \leq r \leq \ell m$, let $\hat{d}_r$ denote the r^{th} higher weight of $\widehat{C}_{\det}(1; \ell, m)$. Then for $r = m+1, \ldots, \ell m$,*

$$\hat{d}_r \geq q^{\ell+m-r-1}\left(\frac{q^r-1}{q-1} + q^{r-2} - 1\right) = \hat{d}_m + q^{\ell+m-r-1}\left(\frac{q^{r-m}-1}{q-1} + q^{r-2} - 1\right),$$

Moreover, equality holds when $r = m+1$ so that $\hat{d}_{m+1} = \hat{d}_m + q^{\ell+m-3}$.

It may be interesting to determine the exact value of *all* the higher weights of not only $\widehat{C}_{\det}(1; \ell, m)$ but also $\widehat{C}_{\det}(t; \ell, m)$ for $1 \leq t \leq \ell$.

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