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Packing of spherical caps of different radii on $\mathbb{S}^2$

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Abstract. We consider the problem of packing spherical caps with (not necessarily) different radii on a three dimensional sphere. We develop two different algorithms for obtaining good constructions as quasi-optimal solutions and comment and compare some results.

1 Introduction

Many extremal problems for arranging points on spheres (dealing with different optimization criteria) actually can be read as packing of spherical caps of equal radius. We take another approach – to consider optimization of packings of caps of different radii. As far as we are aware such problem are extensively studied in two dimensions but quite less attention is paid to the three dimensional case.

The initial motivation for our work was the paper [8] by Müller, Schneider and Schömer where analogous problem is considered in two dimensions (see also [9,10]; further references for the situation in two dimensions is given in [8]).

We firstly applied a simulated-annealing-type algorithm similarly to [8] but later developed a big-bang-type algorithm which appeared to work faster and to give good results as well.

The general problem can be stated as follows.

Problem 1. *Given numbers $r_1, r_2, \ldots, r_n$ (not necessarily distinct) find the smallest radius of a sphere in $\mathbb{R}^3$ that can contain n non-intersecting spherical caps $SC_1, SC_2, \ldots, SC_n$ of radii $r_1, r_2, \ldots, r_n$, respectively.*

Equivalently, one may consider this as follows.

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Problem 2. *Given numbers $r_1, r_2, \ldots, r_n$ (not necessarily distinct) find the largest value of S/S', where S' is the area of a sphere in $\mathbb{R}^3$ that contains n non-intersecting spherical caps $SC_1, SC_2, \ldots, SC_n$ of radii $r_1, r_2, \ldots, r_n$, respectively, and S is the total area of these caps.*

The maximal possible density $\sup S/S'$ is unknown and we are aimed at values close to the constant $\pi/\sqrt{12}$ which is the maximum of the packing of equal circles in the plane. This is heuristic approach and can be implemented similarly with other constants. There are upper bounds by Florian [4, 5] (for a few distinct radii) and de Laat, Filho and Vallentin [6] but the comparison is quite difficult.

We also compute the Riesz energy of our final configurations for certain parameters.

2 Simulated-annealing-type algorithm

The general frame of our first algorithm is analogous to other simulated annealing (SA) algorithms. SA deals with the optimization problem by generating numerous different temporary configurations by gradually modifying the initial one, then chooses new configuration, not always selecting the best one. (The last prevents too quick descend into a local extremum.) It should be also noted that SA is a non-deterministic algorithm, because it is based on random movements in the configuration. Thus the results are expected to vary in different runs (but good implementations would give results within small ranges).

The changes of the configurations are decided according to parameter which we call p-energy of the system. The p-energy is counterpart of the thermal energy in the physical process of annealing. We obtain good packings when the p-energy is minimized.

The major difficulties in such algorithms come from two general requirements: the initial temperature should be high enough and the decreasing must be as slow as necessary to avoid the traps of the local extrema. Different optimization problems require different approaches and adjustments during the implementation. In particular, there is no general rules for the choice of the moves and their number.

We start with a list of the centers of the spherical caps in spherical coordinates and their radii. We need initial large enough sphere which will be decreased later together with the decreasing of the temperature. The idea is to reach simultaneously (with the same number of steps) the zero temperature and the minimum radius. In the beginning we place all caps arbitrarily. Since the initial sphere is large, one does not expect overlaps at this stage.

The temperature frame is given by the initial temperature and the freezing (final) temperature. At every step we decrease the temperature and the radius. Then we calculate the p-energy of the present configuration to decide the change, if any.

For every two distinct caps SC_i and SC_j we define their penalty function (to prevent the caps from intersecting) $P(i,j)$ and then the p-energy of the system

$$H = R + \sum_{i<j} P(i,j),$$

where R is the current radius of the large sphere. Ideally, the optimization stops when R is minimized and $P(i,j) = 0$ for every pair of spherical caps.

The main body of our algorithm includes four procedures (cycles). Two of them (called outer) take care for decreasing the radius of the outer sphere and the decreasing of the temperature. Other two cycles (called inner) perform measurements at each temperature step and try and perform moves inside the current configuration.

Different applications of SA assume different choices for the number of the measurements (iterations) at each temperature step. Our choice went to dynamic numbers. At every iteration we the movement cycle is performed and then the new configuration must be evaluated and accepted or rejected as end product of the iteration. We discard the first 25% of the configurations to allow to the system to equilibrate at the current temperature[2].

Our number of iterations is significantly less than the corresponding number in [8] and this is going to be compensated by larger number of the moves. In fact, these numbers and their dynamic nature are the main difference to the SA implementation from [8].

We apply four types of moves, called Exchange, Short, Leap and Neighbour, and taken with certain probabilities. The move Exchange takes random cap SC_i, finds a (random, if more than one) cap SC_j, $j \neq i$, which minimizes $|r_i - r_j|$, and exchanges SC_i and SC_j. The move Short takes random cap SC_i and adds some small random values to the azimuth and the inclination, i.e. implements relatively short move of SC_i. The move Leap is similar but the changing values are chosen to be relatively large. Finally, the move Neighbour takes a random cap SC_i, finds all its "close" neighbours, chooses randomly one of them SC_j and exchanges SC_i and SC_j.

Once the move is chosen, we perform several checks for its admissibility and accept it if the checks are passed. Otherwise we choose (possibly) new move and continue in the same way. After acceptance and the implementation of the move we calculate the new p-energy and compare it to the current one. According to the general principles of the SA the new configuration is accepted if the new p-energy is better or with certain probability it it is worse.

Sometimes at the end of the work we need to apply some procedure called Afterburner. This cleans up overlaps (if any) to find the final packing.

[2]This is typical for most SA algorithms.

3 Big-bang-type algorithm

As the name suggests the Big-bang-type algorithms work in some sense opposite to the SA algorithms. So our description below is shorter.

Our implementation starts with a small sphere which is "bent" to coincide with the largest cap to be packed. All other caps are placed arbitrarily on the initial sphere. Two outer cycles increase the radius of the sphere and the temperature, respectively, in certain relations. The two inner cycles take care for the iterations and moves, respectively. We prefer again dynamic numbers of the iterations and the moves. There are again four different moves – Swap (changes two caps), Short (moves some cap by short distance), Leap (like Short but by all possible angular distances) and Neighbour (exchanges two caps which are neighbours in certain sense).

After every move we calculate some kind of energy which we call q-energy,

$$H = 2\pi R_0 + \sum_{i<j} \eta_{i,j} k_{i,j},$$

where $k_{i,j} = r_i + r_j - d_{i,j}$, $d_{i,j}$ is the distance between the centers of the caps SC_i and SC_j and $\eta_{i,j} = 0$ or 1 iff $k_{i,j} \leq 0$ or $k_{i,j} > 0$, respectively. The q-energy has to be maximized and according to this aim the new configuration is accepted directly if the new q-energy is larger and with certain small probability otherwise. The latter is done with slightly different (with respect to SA) decision-makers.

4 Some results

We applied both algorithms for many sets of radii for $n = 5$ and some other values of n. The final results coincide in almost all cases and usually give what one could expect as extremal configurations. The densities achieved vary from 81 to 93 per cent.

It is worth to note that our implementations of the BB algorithm appeared to be quite faster than these of the SA algorithm. Since the results are close (usually the same) we decided to switch completely to BB implementations.

We also analyzed the possible connections between the density of the obtained packings and their Riesz energy. The Riesz energy of a spherical code $C \subset \mathbb{S}^{n-1}$ is defined by

$$W(C) = \sum_{x,y \in C, x \neq y} \frac{1}{[d(x,y)]^s},$$

where $s > 0$ is a real parameter. The classical problem (cf. [1–3] and references therein) asks for minimization of the energy for fixed dimension n, degree s and cardinality of the code C.

Our calculations show that better packings have worse (i.e. larger) energy. For example, the best packing we found for spherical caps of radii sets $\{1,1,1,1,5\}$ and $\{1,1,1,2,5\}$ (after re-scaling, of course) have density approx. 92.8 and 91.1 per cent but the corresponding energies are approx. 17.16 and 15.46 (for $s=2$), respectively. It is well-known open problem to find the minimum energy of a 5-point code on $\mathbb{S}^2$. The conjectured optimal code for some wide range of s is the by-pyramid with energy 8.50 for $s=2$ (the best known lower bound in this case is 8.375).

Another possible comparison and analysis of our results come from the possibility to extend plane arrangements to $\mathbb{S}^2$ by using stereographic projections in different ways.

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