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Three-dimensional array codes correcting $2 \times 2 \times 2$-clusters of errors

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Abstract. Linear binary three–dimensional array codes that can correct three–dimensional clusters (or bursts) of errors are presented. New constructions of three-dimensional $2 \times 2 \times 2$–cluster–error–correcting array codes with excess redundancy $\tilde{r}_{n_1,n_2,n_3}(2,2,2) = 17$ are given.

1 Introduction

There are data transmission and storage systems with multidimensional data structures that suffer from multidimensional clusters of errors. Correction of two- and three-dimensional error clusters is required in holographic storage. Two-dimensional and three-dimensional array codes are very suitable for correcting cluster errors in such data structures. In this paper we consider three-dimensional array codes that can correct three-dimensional $2 \times 2 \times 2$-clusters of errors.

For integers $n_l,\ l = 1, 2, 3$ we consider the linear space $V(n_1, n_2, n_3)$ of all binary three-dimensional $n_1 \times n_2 \times n_3$ arrays. A linear k-dimensional $(k \leq n_1n_2n_3)$ subspace $C(n_1, n_2, n_3)$ of the space $V(n_1, n_2, n_3)$ is called a linear binary three-dimensional array $[n_1 \times n_2 \times n_3,\ k]$ code of size (or area) $n_1 \times n_2 \times n_3$ with k information symbols and $r = n_1n_2n_3 - k$ parity-check symbols. Thus, a codeword of the binary linear three-dimensional array code $C(n_1, n_2, n_3)$ is a three-dimensional array $c = (c_{i,j,h})$ where $c_{i,j,h} = 0, 1$ for $i = 0, 1, \ldots, n_1 - 1;\ j = 0, 1, \ldots, n_2 - 1;\ h = 0, 1, \ldots, n_3 - 1.$

A three-dimensional array $e = (e_{i,j,h})$ of size $n_1 \times n_2 \times n_3$ is called a rectangular cluster of size $b_1 \times b_2 \times b_3$ ($b_1 \times b_2 \times b_3$-cluster) if nonzero components of $e = (e_{i,j,h})$ are confined to a rectangular parallelepiped of size $b_1 \times b_2 \times b_3$.

By analogy with two-dimensional array codes ([1]) for $b_1 \times b_2 \times b_3$-cluster-error-correcting array codes whose sizes $n_1 \times n_2 \times n_3$ are much larger than $b_1 \times b_2 \times b_3$ the criterion of the excess redundancy can be used. We define the excess redundancy of the $b_1 \times b_2 \times b_3$-cluster-error-correcting array $[n_1 \times n_2 \times n_3,\ k]$ code $C(n_1, n_2, n_3)$ as

$$\tilde{r}_{n_1,n_2,n_3}(b_1, b_2, b_3) = \lceil r - \log_2 n_1n_2n_3 \rceil, \tag{1}$$

where $r = n_1 n_2 n_3 - k$ is the redundancy of the code $C(n_1, n_2, n_3)$ and $\lceil x \rceil$ is the least integer equal or more than $x \geq 0$. Now if $n = n_1 n_2 n_3$ and $n \to \infty$ we can define for a class $\tilde{C}$ of codes $C(n_1, n_2, n_3)$ the excess redundancy

$$\tilde{r}_C(b_1, b_2, b_3) = \lim_{n\to\infty} (r - \log_2 n_1 n_2 n_3), \tag{2}$$

if such limit exists. If this function is unbounded, we take $\tilde{r}_C(b_1, b_2, b_3) = \infty$.

For any class of three-dimensional $b_1 \times b_2 \times b_3$-cluster-error-correcting array codes

$$\tilde{r}_C(b_1, b_2, b_3) \ \geq \ b_1 b_2 b_3 - 1. \tag{3}$$

For $b_i = 2, \ i = 1, 2, 3$ the excess redundancy

$$\tilde{r}_C(2, 2, 2) > 7. \tag{4}$$

In [2] we had shown that there exist linear three-dimensional $b_1 \times b_2 \times b_3$-cluster-error-correcting array codes with small excess redundancy $\tilde{r}_C(b_1, b_2, b_3) > 3b_1 b_2 b_3 - 5$ for all b_1, b_2, b_3. Constructions of the codes are based on the constructions [3] and used the approaches [4], [5].

In this paper we give new constructions of linear three-dimensional $2\times2\times2$-cluster-error-correcting array codes with the excess redundancy $\tilde{r}_C(2, 2, 2) = 17$.

2 Three-dimensional interleaved array codes

For constructing linear binary three-dimensional array $2 \times 2 \times 2$-cluster-error-correcting codes with small excess redundancy we can use one-dimensional 2-burst-error-correcting codes with the same property.

Let $C(n_1, n_2, n_3)$ be a binary linear three-dimensional array code such that for any code array word $c = (c_{i,j,h})$ we have

$$\sum_{j=0}^{n_2-1} \sum_{h=0}^{n_3-1} c_{i,j,h} = 0, \quad \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} c_{i,j,h} = 0, \quad \sum_{i=0}^{n_1-1} \sum_{h=0}^{n_3-1} c_{i,j,h} = 0.$$

The code $C(n_1, n_2, n_3)$ is the single-error-correcting-double-error-detecting (SEC-DED) array $[n_1 \times n_2 \times n_3, k]$ code with $k = n_1 n_2 n_3 - n_1 - n_2 - n_3 + 2$ information symbols [6].

The parity-check symbols of the code $C(n_1, n_2, n_3)$ are $c_{0,j,h}$, $c_{i,0,h}$ and $c_{i,j,0}$ where $i = 0, \ldots, n_1 - 1$, $j = 0, \ldots, n_2 - 1$, $h = 0, \ldots, n_3 - 1$.

Given the three-dimensional SEC–DED array $[n_1 \times n_2 \times n_3, k]$ code $C(n_1, n_2, n_3)$ we can construct the three-dimensional single-cluster-error-correcting-double-cluster-error-detecting
(SCEC-DCED) array $[b_1 n_1 \times b_2 n_2 \times b_3 n_3, b_1 b_2 b_3 k]$ code $Z(b_1 n_1, b_2 n_2, b_3 n_3)$ by the rectangular three-dimensional interleaving.

The parity-check symbols of the code $Z(b_1 n_1, b_2 n_2, b_3 n_3)$ are $z_{i',j,h}, z_{i,j',h}$ and $z_{i,j,h'}$ where $i = 0, \ldots, b_1(n_1 - 1)$, $j = 0, \ldots, b_2(n_2 - 1)$, $h = 0, \ldots, b_3(n_3 - 1)$, $i' = 0, \ldots, b_1 - 1$, $j' = 0, \ldots, b_2 - 1$, $h' = 0, \ldots, b_3 - 1$. The sets of parity-check symbols $z_{i',j,h}$, $z_{i,j',h}$ and $z_{i,j,h'}$ are confined to rectangular parallelepipeds $(z_{i',j,h})$, $(z_{i,j',h})$ and $(z_{i,j,h'})$ of size $b_1 \times n_2 \times n_3$, $n_1 \times b_2 \times n_3$, $n_1 \times n_2 \times b_3$, respectively. The intersection of these rectangular parallelepipeds is the rectangular parallelepiped $(z_{i',j',h'})$ of size $b_1 \times b_2 \times b_3$.

3 Constructions of linear array codes correcting 2×2- and $2 \times 2 \times 2$-clusters of errors

At first we consider the two-dimensional construction.

Let

$$c = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,2n_2-2} & c_{0,2n_2-1} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,2n_2-2} & c_{1,2n_2-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{2n_1-2,0} & c_{2n_1-2,1} & \cdots & c_{2n_1-2,2n_2-2} & c_{2n_1-2,2n_2-1} \\ c_{2n_1-1,0} & c_{2n_1-1,1} & \cdots & c_{2n_1-1,2n_2-2} & c_{2n_1-1,2n_2-1} \end{bmatrix}$$

be a codeword of the binary two-dimensional array code $C(2n_1, 2n_2)$ constructed by the rectangular three-dimensional interleaving of the two-dimensional array code $C(n_1, n_2)$. If the code $C(n_1, n_2)$ is the direct product of two simple parity-check codes C_1 and C_2, then the code $C(l_1, l_2)$, where $l_1 = 2n_1 - 1$ or $2n_1$, $l_2 = 2n_2 - 1$ or $2n_2$, can correct 2×2-clusters of errors.

Let U and V be one-dimensional cyclic (or shortened cyclic) 2-burst-error-correcting $[l_1, k_1]$ and $[l_2, k_2]$ codes over $GF(2^2)$, respectively.

Let $u = (u_0, u_1, \ldots, u_{l_1-1})$ and $v = (v_0, v_1, \ldots, v_{l_2-1})$ be code words of codes U and V, $u_i = (u_i^{(1)}, u_i^{(2)})$ and $v_j = (v_j^{(1)}, v_j^{(2)})$ be binary representations of elements $u_i, v_j \in GF(2^2)$. Suppose that $\sum_{i=0}^{l_1-1} u_i^{(1)} = \sum_{i=0}^{l_1-1} u_i^{(2)} = 0$ for all $u \in U$.

Let $z = (z_{i,j})$ be the $l_1 \times l_2$ array such that

$z_{i,j} = c_{i,j}$ for $i = 0, \ldots, l_1 - 3$, $j = 1, \ldots, l_2 - 3$,

$z_{i,l_2-2} = c_{i,l_2-2} + u_i^{(1)}$ for $i = 1, \ldots, l_1 - 3$,

$$z_{i,l_2-1} = c_{i,l_2-2} + u_i^{(2)} \text{ for } i = 1, \ldots, l_1 - 3,$$
$$z_{l_1-2,j} = c_{l_1-2,j} + v_j^{(1)} \text{ for } j = 1, \ldots, l_2 - 3,$$
$$z_{l_1-1,j} = c_{l_1-1,j} + v_j^{(2)} \text{ for } j = 1, \ldots, l_2 - 3,$$
$$z_{l_1-2,l_2-2} = c_{l_1-2,l_2-2} + u_{l_1-2}^{(1)} + v_{l_2-2}^{(1)}, \; z_{l_1-1,l_2-2} = c_{l_1-1,l_2-2} + u_{l_1-2}^{(2)} + v_{l_2-2}^{(1)}$$
$$z_{l_1-2,l_2-1} = c_{l_1-2,l_2-1} + u_{l_1-2}^{(2)} + v_{l_2-1}^{(2)}, \; z_{l_1-1,l_2-1} = c_{l_1-1,l_2-1} + u_{l_1-2}^{(2)} + v_{l_2-1}^{(2)}.$$

Lemma 1 *The set of arrays* $(z_{i,j})$ *is a binary linear two-dimensional array* 2×2*–cluster–error–correcting* $[l_1 \times l_2, l_1 l_2 - 2(l_1 - k_1) - 2(l_2 - k_2) + 4]$ *code* $Z(l_1, l_2)$.

We can use one-dimensional cyclic 2–burst–error–correcting Fire codes over $GF(2^2)$ as codes U and V. Furthermore we can use the following lemma.

Lemma 2 *Let* $p(x)$ *be a irreducible polynomial of degree* m *over* $GF(2^c)$. *Then the polynomial* $g(x) = (x^2 + 1)p(x)$ *generate the one-dimensional cyclic* 2*–burst–error–correcting code over* $GF(2^c)$ *of length* $\frac{2(2^{cm}-1)}{2^c-1}$.

Example. Let m_1, m_2 be positive integers and $l_1 = \frac{2(2^{2m_1}-1)}{3}$, $l_2 = \frac{2(2^{2m_2}-1)}{3}$. Let U and V be cyclic $[l_1, k_1]$ and $[l_2, k_2]$ codes over $GF(2^2)$, satisfying lemma 2. Then the binary two-dimensional array code $Z(l_1, l_2)$ of size $l_1 \times l_2$ with $r = 2m_1 + 2m_2 + 4$ parity–check symbols can correct single error clusters of size 2×2. The excess redundancy of the code $Z(l_1, l_2)$ is

$$\tilde{r}_{n_1,n_2}(2, 2) = 6.$$

Now we consider the construction of three-dimensional array $2\times2\times2$-cluster-error-correcting codes. For constructing the linear binary three-dimensional array codes with small excess redundancy we use one-dimensional cyclic (or shortened cyclic) 2-burst-error-correcting codes over $GF(2^4)$ with the same property.

Let α be a primitive element of the Galois field $GF(2^4)$. There is a one-to-one correspondence between elements α^i and binary arrays of size $GF(2^4)$. Therefore we can represent the cyclic 2-burst-error-correcting $[l, k]$ code L over $GF(2^4)$ as the binary three-dimensional array $[l \times 2 \times 2, 4k]$ code U_L correcting $2 \times 2 \times 2$-clusters of errors. A code word of the code U_L is the the rectangular parallelepiped $(u_{i,j,h})$ of size $l \times 2 \times 2$.

Theorem 1 *Let* $C(l_1, l_2, l_3)$ *be the binary linear three-dimensional twice interleaved array code correcting* $2 \times 2 \times 2$*-clusters of errors. Let* L_i, L_j *and* L_h *be the cyclic* 2*-burst-error-correcting* $[l_{j,h}, k_{j,h}]$, $[l_{h,i}, k_{h,i}]$ *and* $[l_{i,j}, k_{i,j}]$ *codes over*

$GF(2^4)$, satisfying Lemma 2, respectively. Then the sum of the codes $C(l_1, l_2, l_3)$ and U_{L_i}, U_{L_j}, U_{L_h} is the binary linear three-dimensional array code $Z(l_1, l_2, l_3)$ of size $l_1 \times l_2 \times l_3$ with $r = 4(l_{i,j} - k_{i,j}) + 4(l_{j,h} - k_{j,h}) + 4(l_{h,i} - k_{h,i}) - 16$ parity-check symbols that can correct single error clusters of size $2 \times 2 \times 2$.

The excess redundancy of the code $Z(l_1, l_2, l_3)$ is

$$\tilde{r}_{l_1,l_2,l_3}(2, 2, 2) = 17.$$

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