Three-dimensional array codes correcting $2 \times 2 \times 2$-clusters of errors

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Abstract. Linear binary three-dimensional array codes that can correct three-dimensional clusters (or bursts) of errors are presented. New constructions of three-dimensional $2 \times 2 \times 2$-cluster-error-correcting array codes with excess redundancy $\tilde{r}_{n_1,n_2,n_3}(2,2,2) = 17$ are given.

1 Introduction

There are data transmission and storage systems with multidimensional data structures that suffer from multidimensional clusters of errors. Correction of two- and three-dimensional error clusters is required in holographic storage. Two-dimensional and three-dimensional array codes are very suitable for correcting cluster errors in such data structures. In this paper we consider three-dimensional array codes that can correct three-dimensional $2 \times 2 \times 2$-clusters of errors.

For integers $n_l$, $l = 1, 2, 3$ we consider the linear space $V(n_1,n_2,n_3)$ of all binary three-dimensional $n_1 \times n_2 \times n_3$ arrays. A linear $k$-dimensional $(k \leq n_1n_2n_3)$ subspace $C(n_1,n_2,n_3)$ of the space $V(n_1,n_2,n_3)$ is called a linear binary three-dimensional array $[n_1 \times n_2 \times n_3, k]$ code of size (or area) $n_1 \times n_2 \times n_3$ with $k$ information symbols and $r = n_1n_2n_3 - k$ parity-check symbols. Thus, a codeword of the binary linear three-dimensional array code $C(n_1,n_2,n_3)$ is a three-dimensional array $c = (c_{i,j,h})$ where $c_{i,j,h} = 0, 1$ for $i = 0, 1, \ldots, n_1 - 1$; $j = 0, 1, \ldots, n_2 - 1$; $h = 0, 1, \ldots, n_3 - 1$.

A three-dimensional array $e = (e_{i,j,h})$ of size $n_1 \times n_2 \times n_3$ is called a rectangular cluster of size $b_1 \times b_2 \times b_3$ ($b_1 \times b_2 \times b_3$-cluster) if nonzero components of $e = (e_{i,j,h})$ are confined to a rectangular parallelepiped of size $b_1 \times b_2 \times b_3$.

By analogy with two-dimensional array codes ([1]) for $b_1 \times b_2 \times b_3$-cluster-error-correcting array codes whose sizes $n_1 \times n_2 \times n_3$ are much larger than $b_1 \times b_2 \times b_3$ the criterion of the excess redundancy can be used. We define the excess redundancy of the $b_1 \times b_2 \times b_3$-cluster-error-correcting array $[n_1 \times n_2 \times n_3, k]$ code $C(n_1,n_2,n_3)$ as

$$\tilde{r}_{n_1,n_2,n_3}(b_1,b_2,b_3) = \lceil r - \log_2 n_1n_2n_3 \rceil,$$  

(1)
where \( r = n_1n_2n_3 - k \) is the redundancy of the code \( C(n_1, n_2, n_3) \) and \( \lceil x \rceil \) is the least integer equal or more than \( x \geq 0 \). Now if \( n = n_1n_2n_3 \) and \( n \to \infty \) we can define for a class \( \tilde{C} \) of codes \( C(n_1, n_2, n_3) \) the excess redundancy

\[
\tilde{r}_C(b_1, b_2, b_3) = \lim_{n \to \infty} (r - \log_2 n_1n_2n_3),
\]

if such limit exists. If this function is unbounded, we take \( \tilde{r}_C(b_1, b_2, b_3) = \infty \).

For any class of three-dimensional \( b_1 \times b_2 \times b_3 \)-cluster-error-correcting array codes

\[
\tilde{r}_C(b_1, b_2, b_3) \geq b_1b_2b_3 - 1.
\]

For \( b_i = 2, i = 1, 2, 3 \) the excess redundancy

\[
\tilde{r}_C(2, 2, 2) > 7.
\]

In [2] we had shown that there exist linear three-dimensional \( b_1 \times b_2 \times b_3 \)-cluster-error-correcting array codes with small excess redundancy \( \tilde{r}_C(b_1, b_2, b_3) > 3b_1b_2b_3 - 5 \) for all \( b_1, b_2, b_3 \). Constructions of the codes are based on the constructions [3] and used the approaches [4], [5].

In this paper we give new constructions of linear three-dimensional \( 2 \times 2 \times 2 \)-cluster-error-correcting array codes with the excess redundancy \( \tilde{r}_C(2, 2, 2) = 17 \).

## 2 Three-dimensional interleaved array codes

For constructing linear binary three-dimensional array \( 2 \times 2 \times 2 \)-cluster-error-correcting codes with small excess redundancy we can use one-dimensional \( 2 \)-burst-error-correcting codes with the same property.

Let \( C(n_1, n_2, n_3) \) be a binary linear three-dimensional array code such that for any code array word \( c = (c_{i,j,h}) \) we have

\[
\sum_{j=0}^{n_2-1} \sum_{h=0}^{n_3-1} c_{i,j,h} = 0, \quad \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} c_{i,j,h} = 0, \quad \sum_{i=0}^{n_1-1} \sum_{h=0}^{n_3-1} c_{i,j,h} = 0.
\]

The code \( C(n_1, n_2, n_3) \) is the single-error-correcting-double-error-detecting (SEC-DED) array \( [n_1 \times n_2 \times n_3, k] \) code with \( k = n_1n_2n_3 - n_1 - n_2 - n_3 + 2 \) information symbols [6].

The parity-check symbols of the code \( C(n_1, n_2, n_3) \) are \( c_{0,j,h} \), \( c_{i,0,h} \) and \( c_{i,j,0} \) where \( i = 0, \ldots, n_1 - 1 \), \( j = 0, \ldots, n_2 - 1 \), \( h = 0, \ldots, n_3 - 1 \).
Given the three-dimensional SEC–DED array \([n_1 \times n_2 \times n_3, k]\) code \(C(n_1, n_2, n_3)\) we can construct the three-dimensional single-cluster-error-correcting-double-cluster-error-detecting (SCEC-DCED) array \([b_1 n_1 \times b_2 n_2 \times b_3 n_3, b_1 b_2 b_3 k]\) code \(Z(b_1 n_1, b_2 n_2, b_3 n_3)\) by the rectangular three-dimensional interleaving.

The parity-check symbols of the code \(Z(b_1 n_1, b_2 n_2, b_3 n_3)\) are \(z_{i,j,h}, z_{i,j',h}\) and \(z_{i,j,h'}\) where \(i = 0, \ldots, b_1(n_1 - 1),\) \(j = 0, \ldots, b_2(n_2 - 1),\) \(h = 0, \ldots, b_3(n_3 - 1),\) \(i' = 0, \ldots, b_1 - 1,\) \(j' = 0, \ldots, b_2 - 1,\) \(h' = 0, \ldots, b_3 - 1.\)

The sets of parity-check symbols \(z_{i,j,h}, z_{i,j',h}\) and \(z_{i,j,h'}\) are confined to rectangular parallelepipeds \((z_{i,j,h}), (z_{i,j',h})\) and \((z_{i,j,h'})\) of size \(b_1 \times n_2 \times n_3, n_1 \times b_2 \times n_3, n_1 \times n_2 \times b_3,\) respectively. The intersection of these rectangular parallelepipeds is the rectangular parallelepiped \((z_{i,j',h'})\) of size \(b_1 \times b_2 \times b_3,\)

### 3 Constructions of linear array codes correcting \(2 \times 2\)- and \(2 \times 2 \times 2\)-clusters of errors

At first we consider the two-dimensional construction.

Let \(C = \begin{bmatrix} 
  c_{0,0} & c_{0,1} & \ldots & c_{0,2n_2-2} & c_{0,2n_2-1} \\
  c_{1,0} & c_{1,1} & \ldots & c_{1,2n_2-2} & c_{1,2n_2-1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  c_{2n_1-2,0} & c_{2n_1-2,1} & \ldots & c_{2n_1-2,2n_2-2} & c_{2n_1-2,2n_2-1} \\
  c_{2n_1-1,0} & c_{2n_1-1,1} & \ldots & c_{2n_1-1,2n_2-2} & c_{2n_1-1,2n_2-1} 
\end{bmatrix}\)

be a codeword of the binary two-dimensional array code \(C(2n_1, 2n_2)\) constructed by the rectangular three-dimensional interleaving of the two-dimensional array code \(C(n_1, n_2)\). If the code \(C(n_1, n_2)\) is the direct product of two simple parity-check codes \(C_1\) and \(C_2\), then the code \(C(l_1, l_2)\), where \(l_1 = 2n_1 - 1\) or \(2n_1,\) \(l_2 = 2n_2 - 1\) or \(2n_2,\) can correct \(2 \times 2\)-clusters of errors.

Let \(U\) and \(V\) be one-dimensional cyclic (or shortened cyclic) \(2\)-burst-error-correcting \([l_1, k_1]\) and \([l_2, k_2]\) codes over \(GF(2^2)\), respectively.

Let \(u = (u_0, u_1, \ldots, u_{l_1-1})\) and \(v = (v_0, v_1, \ldots, v_{l_2-1})\) be code words of codes \(U\) and \(V\), \(u_i = (u_i^{(1)}, u_i^{(2)})\) and \(v_j = (v_j^{(1)}, v_j^{(2)})\) be binary representations of elements \(u_i, v_j \in GF(2^2)\). Suppose that \(\sum_{i=0}^{l_1-1} u_i^{(1)} = \sum_{i=0}^{l_1-1} u_i^{(2)} = 0\) for all \(u \in U\).

Let \(z = (z_{i,j})\) be the \(l_1 \times l_2\) array such that
\[
  z_{i,j} = c_{i,j} \quad \text{for} \quad i = 0, \ldots, l_1 - 3, \quad j = 1, \ldots, l_2 - 3,
\]
\[
  z_{i,l_2-2} = c_{i,l_2-2} + u_i^{(1)} \quad \text{for} \quad i = 1, \ldots, l_1 - 3,
\]
Lemma 1 The set of arrays \((z_{i,j})\) is a binary linear two-dimensional array \(2 \times 2\)-cluster-error-correcting \([l_1 \times l_2, l_1 l_2 - 2(l_1 - k_1) - 2(l_2 - k_2) + 4]\) code \(Z(l_1, l_2)\).

We can use one-dimensional cyclic 2-burst-error-correcting Fire codes over \(GF(2^4)\) as codes \(U\) and \(V\). Furthermore we can use the following lemma.

Lemma 2 Let \(p(x)\) be a irreducible polynomial of degree \(m\) over \(GF(2^e)\). Then the polynomial \(g(x) = (x^2 + 1)p(x)\) generate the one-dimensional cyclic 2-burst-error-correcting code over \(GF(2^e)\) of length \(\frac{2(2^{2m_e} - 1) - 1}{2^{e-1}}\).

Example. Let \(m_1, m_2\) be positive integers and \(l_1 = \frac{2(2^{2m_1} - 1)}{3}\), \(l_2 = \frac{2(2^{2m_2} - 1)}{3}\). Let \(U\) and \(V\) be cyclic \([l_1, k_1]\) and \([l_2, k_2]\) codes over \(GF(2^e)\), satisfying lemma 2. Then the binary two-dimensional array code \(Z(l_1, l_2)\) of size \(l_1 \times l_2\) with \(r = 2m_1 + 2m_2 + 4\) parity-check symbols can correct single error clusters of size \(2 \times 2\). The excess redundancy of the code \(Z(l_1, l_2)\) is

\[r = 6\]

Now we consider the construction of three-dimensional array \(2 \times 2 \times 2\)-cluster-error-correcting codes. For constructing the linear binary three-dimensional array codes with small excess redundancy we use one-dimensional cyclic (or shortened cyclic) 2-burst-error-correcting codes over \(GF(2^4)\) with the same property.

Let \(\alpha\) be a primitive element of the Galois field \(GF(2^4)\). There is a one-to-one correspondence between elements \(\alpha^i\) and binary arrays of size \(GF(2^4)\). Therefore we can represent the cyclic 2-burst-error-correcting \([l, k]\) code \(L\) over \(GF(2^4)\) as the binary three-dimensional array \([l \times 2 \times 2, 4k]\) code \(U_L\) correcting \(2 \times 2 \times 2\)-clusters of errors. A code word of the code \(U_L\) is the rectangular parallelepiped \((u_{i,j,k})\) of size \(l \times 2 \times 2\).

Theorem 1 Let \(C(l_1, l_2, l_3)\) be the binary linear three-dimensional twice interleaved array code correcting \(2 \times 2 \times 2\)-clusters of errors. Let \(L_i, L_j\) and \(L_h\) be the cyclic 2-burst-error-correcting \([l, k_i, l_j], [l, k_j, l_h], [l, k_i, l_j, l_h]\) codes over
GF(2^4), satisfying Lemma 2, respectively. Then the sum of the codes $C(l_1, l_2, l_3)$ and $U_{L_1}, U_{L_2}, U_{L_3}$ is the binary linear three-dimensional array code $Z(l_1, l_2, l_3)$ of size $l_1 \times l_2 \times l_3$ with $r = 4(l_{i,j} - k_{i,j}) + 4(l_{j,h} - k_{j,h}) + 4(l_{h,i} - k_{h,i}) - 16$ parity-check symbols that can correct single error clusters of size $2 \times 2 \times 2$.

The excess redundancy of the code $Z(l_1, l_2, l_3)$ is

$$\tilde{r}_{l_1,l_2,l_3}(2,2,2) = 17.$$  

References


