Eleventh International Workshop on Algebraic and Combinatorial Coding Theory
June 16-22, 2008, Pamporovo, Bulgaria *pp. 35–39*

Nonexistence results for spherical 7-designs

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Abstract. We obtain new nonexistence results for spherical 7-designs of odd cardinality. Our approach continues similar investigations for smaller strengths. We combine polynomial techniques with some geometric argument to obtain restrictions of the structure of 7-designs with fixed cardinality.

1 Introduction

A spherical τ-design [2] is a spherical code $C \subset \mathbf{S}^{n-1}$ such that for every point $y \in \mathbf{S}^{n-1}$ and for every real polynomial $f(t)$ of degree at most τ, the equality

$$\sum_{x \in C} f(\langle x, y \rangle) = f_0 |C|. \tag{1}$$

holds, where f_0 is the first coefficient in the expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ in terms of the Gegenbauer polynomials [1, Chapter 22]. The number τ is called strength of C. When $y \in C$, (1) becomes

$$\sum_{x \in C \setminus \{y\}} f(\langle t_i(x) \rangle) = f_0 |C| - f(1), \tag{2}$$

where $t_1(x) \leq t_2(x) \leq \cdots \leq t_{|C|-1}(x)$ are the inner products of $x \in C$ with all other points of C.

Polynomial techniques use suitable polynomials in (1) and (2) for obtaining bounds on some inner products. Restrictions on the structure of spherical designs via polynomial techniques were described in 1997 by Fazekas-Levenshtein [8] (see also [9]) and proved to work for nonexistence results by Boyvalenkov-Danev-Nikova [6] (see also [3, 4, 5]). In this paper we continue investigations from [5] by obtaining new nonexistence results for 7-designs in dimensions $n \leq 20$.

2 Preliminaries

Let $C \subset \mathbf{S}^{n-1}$ be a 7-design. Then

$$|C| \geq 2\binom{n+2}{3} = \frac{n(n+1)(n+2)}{3} \tag{3}$$

by the Delsarte-Goethals-Seidel bound [2].

We use some results and notations from [3, 4, 5, 8, 9]. The parameters α_i are roots of certain Jacobi polynomials and the definition of the weight ρ_0 can be found in [9].

Lemma 1. [3] *Let $C \subset \mathbb{S}^{n-1}$ be a τ-design with odd $\tau = 2e-1$. Then for every point $x \in C$ we have $t_1(x) \leq \alpha_0$ and $t_{|C|-1}(x) \geq \alpha_{e-1}$. In particular, we have $s(C) \geq \alpha_{e-1}$. If $|C|$ is odd then there exist a point $x \in C$ such that $t_2(x) \leq \alpha_0$.*

Lemma 2. [4] *Let $C \subset \mathbb{S}^{n-1}$ be a τ-design with odd $\tau = 2e-1$ and odd cardinality $|C|$. Then there exist three distinct points $x, y, z \in C$ such that $t_1(x) = t_1(y)$ and $t_2(x) = t_1(z)$. Moreover, we have $t_{|C|-1}(z) \geq \max\{\alpha_{e-1}, 2\alpha_0^2 - 1\}$. In particular, we have $s(C) \geq \max\{\alpha_{e-1}, 2\alpha_0^2 - 1\}$.*

Theorem 3. [3] *If $C \subset \mathbb{S}^{n-1}$ is a τ-design with odd $\tau = 2e-1$ and odd $|C|$ then $\rho_0|C| \geq 2$.*

It is proved in [5] that the necessary condition $\rho_0|C| \geq 2$ can be replaced by the stronger $\rho_0|C| \geq 3$ for 3-designs (with a few exceptions) in dimensions $8 \leq n \leq 50$ and for 5-designs in dimensions $5 \leq n \leq 25$. In this paper we prove that $\rho_0|C| \geq 3$ is necessary for 7-designs of odd cardinalities in dimensions $5 \leq n \leq 20$. Moreover, we obtain nonexistence results in several cases where $\rho_0|C| > 3$.

It is convenient to use the following notation: $U_{\tau,i}(x)$ (respectively $L_{\tau,i}(x)$) for any upper (resp. lower) bound on the inner product $t_i(x)$. When a bound does not depend on x we omit x in the notation. For example, the first bound from Lemma 1 is $t_1(x) \leq U_{\tau,1} = \alpha_0$ and the last bound from Lemma 2 is $t_{|C|-1}(z) \geq L_{\tau,|C|-1}(z) = \max\{\alpha_{e-1}, 2\alpha_0^2 - 1\}$.

3 Brief description of the algorithm

Assuming the existence of a 7-design on $\mathbf{S}^{n-1}$ with odd $|C|$ and $2\alpha_0^2 - 1 > \alpha_3$, we consider a special triple of points $x, y, z \in C$ as in Lemma 2. We focus on the inner products in $I(z)$. Sometimes we need to consider the point $u \in C$

such that $\langle u,z\rangle = t_2(z)$. We consecutively obtain bounds $L_{7,1}(z) \leq t_1(z)$, $t_2(z) \leq U_{7,2}(z)$ and $L_{7,3}(z) \leq t_3(z)$ using suitable polynomials.

Sometimes we get contradictions at the beginning – already from the first bounds $t_1(z) \geq L_{7,1}(z) > U_{7,1}(z) = \alpha_0 \geq t_1(z)$ (this happens in cases where $\rho_0|C|$ is close from above to 2) and further by $t_2(z) \leq U_{7,2}(z) < L_{7,1}(z) \leq t_1(z)$. When $U_{7,2}(z) \geq L_{7,1}(z)$, we consider two cases for the location of $t_2(z)$ with respect to α_0.

Case 1. If $t_2(z) \in [\alpha_0, U_{7,2}(z)]$ (this can happen only when $\alpha_0 \leq U_{7,2}(z)$) then we obtain new upper bound $t_1(z) \leq U_{7,1}(z) < \alpha_0$ which can be used for obtaining a contradiction. If necessary (in a few cases) we organize an iteration procedure.

Case 2. If $t_2(z) \in [t_1(z), \alpha_0]$, then we consider the point $u \in C$ such that $t_2(z) = \langle z,u\rangle$. It follows from [5, Section 4] that some special quadruple $\{x,y,z,u\} \subset C$ exists such that $\max\{t_{|C|-2}(z), t_{|C|-2}(x)\} \geq 2\alpha_0^2 - 1$. In both cases we continue with new bounds $L_{7,3}(z)$ and $U_{7,1}(z)$ which can be used for obtaining a contradiction. In some case we need more careful consideration of the location of some inner products and iteration procedures.

All symbolic and numerical calculations were performed by MAPLE with high enough precision. All programs and results (symbolic and numerical) are available upon request.

4 The new nonexistence results

After [4], there are 291 open cases in dimensions $3 \leq n \leq 20$, with odd $|C|$ and $2 \leq \rho_0|C| < 3$. In every such case we have $2\alpha_0^2 - 1 > \alpha_3$, i.e. $t_{|C|-1}(z) \geq L_{3,|C|-1}(z) = 2\alpha_0^2 - 1$ by Lemma 2. Applying our algorithm we obtain nonexistence in all cases with only one exception – the case $n = 4$, $|C| = 43$. There are 18 cases of nonexistence with $\rho_0|C| > 3$ as well.

In the table below we give lower bounds on

$$B_{\mathrm{odd}}(n,7) = \min\{|C| : C \subset \mathbf{S}^{n-1} \text{ is a 7-design, } |C| \text{ is odd}\}.$$

The bounds from [7] (the second column in the table) come from pure linear programming and are better than the Delsarte-Goethals-Seidel bound (3) in dimensions 5, 6, and 7 only. No examples in small dimensions are given in [6, 3, 4] but we know that the best bounds come by the method from [4] (the third column in the table). In the fourth column we give the results from the method from Section 3 when $\rho_0|C| \in [2,3)$ and the fifth column gives the results from the method from Section 3 when $\rho_0|C| > 3$. So the best bounds are the last entries in the rows.

Table. Lower bounds on $B_{\text{odd}}(n,7)$, $3 \le n \le 20$.

n	[7]	[4]	This paper $\rho_0\lvert C\rvert \in [2,3)$	This paper $\rho_0\lvert C\rvert \ge 3$
3		23	23	
4		43	43	
5	73	75	77	
6	117	119	123	
7	173	177	183	
8		253	261	
9		347	359	
10		463	477	
11		601	619	621
12		765	789	
13		957	985	987
14		1175	1213	1215
15		1427	1471	1475
16		1713	1767	1769
17		2031	2097	2101
18		2393	2467	2473
19		2791	2879	2885
20		3233	3333	3341

In [6], the asymptotic lower bound $B_{\text{odd}}(n,7) \gtrsim \frac{(1+\sqrt[7]{2})n^3}{6} \approx 0.35068n^3$ was proved. This was obtained again in [3] despite the results in small dimensions from [3] are better than those from [6]. The best known asymptotic lower bound is $B_{\text{odd}}(n,7) \gtrsim 0.35314n^3$ from [4]. The results from this paper suggest that further improvements are possible by our method. However, we still could not overcome the technical difficulties on this way.

Acknowledgment. This research was partially supported by the Bulgarian NSF under Contract MM-1405/2004 and the SF of Sofia University under Contract 171/05.2008.

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