Eleventh International Workshop on Algebraic and Combinatorial Coding Theory
June 16-22, 2008, Pamporovo, Bulgaria *pp. xxx-xxx*

On the Error-Correcting Capabilities of Low-Complexity Decoded LDPC Codes with Constituent Hamming Codes

Victor Zyablov zyablov@iitp.ru
Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow 101447, RUSSIA

Maja Lončar maja@eit.lth.se
Rolf Johannesson rolf@eit.lth.se
Dept. of Electrical and Information Technology, Lund University, P. O. Box 118, SE-22100 Lund, SWEDEN

Pavel Rybin prybin@iitp.ru
Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow 101447, RUSSIA

Abstract. Hamming code-based LDPC (H-LDPC) block codes are obtained by replacing the single parity-check constituent codes in Gallager's LDPC codes with Hamming codes. This paper investigates the asymptotic performance of ensembles of random H-LDPC codes, used over the binary symmetric channel and decoded with a low-complexity hard-decision iterative decoding algorithm. It is shown that there exist H-LDPC codes for which such iterative decoding corrects any error pattern with a number of errors that grows linearly with the code length. The number of required decoding iterations is a logarithmic function of the code length. The fraction of correctable errors is computed numerically for different code parameters.

1 Introduction

Concatenated code structures can yield powerful codes, which achieve good performance with low-complexity decoding, based on using simple constituent decoders as separate modules. A method for constructing long codes from short constituent codes, based on bipartite graphs, was introduced by Tanner in [1]. In this method, one of the two sets of nodes in a bipartite graph is associated with code symbols, while the other set is associated with constituent block codes of length equal to the node degree. These two sets of nodes are hereinafter referred to as variable nodes and constraint nodes, respectively.

This work was supported in part by the Royal Swedish Academy of Sciences in cooperation with the Russian Academy of Sciences and in part by the Swedish Research Council under Grant 621-2007-6281.

Tanner's general code construction unifies many known code families that can be obtained by choosing different underlying bipartite graphs and associating different constituent codes with their constraint nodes. For example, product codes [2], Gallager's Low-Density Parity-Check (LDPC) codes [3], expander codes [4], [5], and woven graph codes [6], [7] can all be described using a bipartite graph-based approach.

For Gallager's LDPC codes [3], each constraint node in the corresponding bipartite graph represents a single parity-check (SPC) code over the variable nodes connected to it. In this case, the parity-check matrix of the code coincides with the adjacency matrix[1] of the corresponding bipartite graph. If the degree of each node is very small compared to the number of variable nodes (code length) the parity-check matrix is sparse. When the bipartite graph is regular, all variable nodes have degree j and all constraint nodes have degree k. Then the parity-check matrix contains j ones in each column and k ones in each row, and it specifies a (j, k)-regular LDPC code.

The error-correcting capabilities of LDPC codes for the binary symmetric channel (BSC) were studied in [8], where it was shown that there exist LDPC codes capable of correcting a portion of errors that grows linearly with the code length n, with decoding complexity $\mathcal{O}(n \log n)$. A similar result for expander codes was proven in [4], [5], [9].

The SPC codes associated with constraint nodes in the Tanner graph of an LDPC code can be replaced with other constituent block codes (*e.g.*, Hamming codes [10], BCH codes [11], or Reed-Solomon codes [12]), which yields alternative constructions of LDPC codes, often referred to as generalized LDPC codes. The parity-check matrix of such an LDPC code is obtained by replacing every 1 in the graph's adjacency matrix with a column of the constituent code's parity-check matrix, and every 0 with an all-zero column.

Hamming code-based LDPC (H-LDPC) codes were first studied in [13]. Distance properties and iterative soft-decision decoding of the H-LDPC codes were further investigated in [10] and [11]. In [14], it was shown that an ensemble of H-LDPC codes contains codes with a minimum distance that asymptotically almost meets the Varshamov-Gilbert bound.

In this paper, we consider the asymptotic performance of random H-LDPC codes, when the code length n grows to infinity. We will prove that there exist H-LDPC codes which, when decoded with a simple iterative decoder of complexity $\mathcal{O}(n \log n)$, can correct any error pattern with a number of errors growing linearly with the code length. Our approach builds upon the work of [8] where such a result was proved for LDPC codes with constituent SPC codes which have minimum distance $d_0 = 2$. A similar result holds for expander codes if the constituent codes have large enough minimum distance, *cf.* [4], [5],

[1]Here it is assumed that the adjacency matrix $\boldsymbol{A}$ of a bipartite graph with two vertex sets $\mathcal{V}_1$ and $\mathcal{V}_2$ is a $|\mathcal{V}_1| \times |\mathcal{V}_2|$ binary matrix specifying connections among vertices, that is, $(\boldsymbol{A})_{ij} = 1$ iff nodes $v_i \in \mathcal{V}_1$ and $v_j \in \mathcal{V}_2$ are connected with a branch.

[9]. The work presented here, with constituent Hamming codes of minimum distance $d_0 = 3$, is a step towards 'closing the gap' between these two results.

2 Construction and Properties of H-LDPC Codes

An (n_0, k_0, d_0) Hamming code has length $n_0 = 2^m - 1$, dimension $k_0 = n_0 - m$, code rate $R_0 = 1 - m/n_0$, and minimum distance $d_0 = 3$, where $m \geq 2$ (for $m = 2$ the code reduces to the length-3 repetition code). Hamming codes are perfect single-error correcting codes, that is, they correct all error patterns with one error, and no others, and their covering radius is equal to $\rho = \lfloor (d_{\min} - 1)/2 \rfloor = 1$.

A parity-check matrix $\boldsymbol{H}_0$ of a Hamming code is an $m \times n_0$ matrix whose columns are all nonzero binary m-tuples. We will consider H-LDPC codes with identical constituent Hamming codes. Let $\boldsymbol{H}_\mathrm{b}$ denote a block-diagonal matrix with the b constituent parity-check matrices $\boldsymbol{H}_0$ on the main diagonal, that is,

$$\boldsymbol{H}_\mathrm{b} = \begin{pmatrix} \boldsymbol{H}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{H}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{H}_0 \end{pmatrix} \tag{1}$$

where b is very large. The matrix $\boldsymbol{H}_\mathrm{b}$ is of size $bm \times bn_0$. Let $\pi(\boldsymbol{H}_\mathrm{b})$ denote a random column permutation of $\boldsymbol{H}_\mathrm{b}$. Then the matrix constructed using $\ell \geq 2$ such permutations as *layers*,

$$\boldsymbol{H} = \begin{pmatrix} \boldsymbol{H}_1 \\ \boldsymbol{H}_2 \\ \vdots \\ \boldsymbol{H}_\ell \end{pmatrix} = \begin{pmatrix} \pi_1(\boldsymbol{H}_\mathrm{b}) \\ \pi_2(\boldsymbol{H}_\mathrm{b}) \\ \vdots \\ \pi_\ell(\boldsymbol{H}_\mathrm{b}) \end{pmatrix} \tag{2}$$

is a sparse $\ell bm \times bn_0$ parity-check matrix which characterizes the ensemble of Hamming code-based LDPC codes of length $n = bn_0$, where $n \gg n_0$. Let $\mathscr{C}(n_0, \ell, b)$ denote this ensemble. For a given constituent Hamming code with parity-check matrix $\boldsymbol{H}_0$, the elements of the ensemble $\mathscr{C}(n_0, \ell, b)$ are obtained by sampling independently the permutations π_l, $l = 1, 2, ..., \ell$, which are all equiprobable. The rate of a code $\mathcal{C} \in \mathscr{C}(n_0, \ell, b)$ is lower-bounded by [1]

$$R \geq 1 - \frac{\ell b(n_0 - k_0)}{n} = 1 - \ell(1 - R_0) \tag{3}$$

with equality iff the matrix $\boldsymbol{H}$ has full rank. This imposes a restriction on the rate of the constituent codes, namely,

$$R_0 > 1 - \frac{1}{\ell}$$

that is, the more layers there are, the higher the rate of the constituent codes must be.

The construction defined by (2) is a generalization of Gallager's construction [3] of the LDPC matrices, where the constituent codes are $(n_0, n_0 - 1)$ single parity-check (SPC) codes, for which $\boldsymbol{H}_0 = (1\ 1\ ...\ 1)$. In that case, the matrix $\boldsymbol{H}$ has n_0 ones in each row, and ℓ ones in each column. Such a matrix has density $1/b$, and it specifies an (ℓ, n_0)-regular LDPC code. The parity-check matrix of an H-LDPC code, given by (2), is, in general, irregular.

The H-LDPC codes from the ensemble $\mathscr{C}(n_0, \ell, b)$ contain ℓb constituent Hamming codes; b in each layer. Such H-LDPC codes can be represented by a Tanner graph [1] with $n = bn_0$ variable nodes, and ℓb constraint nodes, as illustrated in Figure 1. Each constraint node comprises $n_0 - k_0$ parity-check constraints specified by the rows of the corresponding constituent parity-check matrix. If a codesymbol is checked by a constituent code (that is, by at least one row of its parity-check matrix), there is a branch connecting the corresponding variable node and the constraint node. Each codesymbol is checked by exactly one Hamming code in each layer. The graph is regular, with the variable-node degree equal to ℓ, and the constraint-node degree equal to n_0. Such a graph is a special type of expander [15], [4], where it is required that the ℓ constraint nodes adjacent to each variable node all belong to different layers.

Consider communication over a binary symmetric channel (BSC) using H-LDPC codes with hard-decision decoding. Let $\boldsymbol{v}$ be the transmitted codeword and $\boldsymbol{e}$ be the error pattern. Then the received sequence is given by $\boldsymbol{r} = \boldsymbol{v} + \boldsymbol{e}$. The weight of the error sequence is $W = |\boldsymbol{e}|$ and the fraction of erroneous symbols is $\omega = W/n$. For code length $n \to \infty$, the fraction of erroneous symbols ω converges in probability to the crossover probability of the BSC.

The syndrome vector computed at the receiver is given by

$$\boldsymbol{s} = \boldsymbol{r}\boldsymbol{H}^{\mathrm{T}} = (\boldsymbol{e}\boldsymbol{H}_1^{\mathrm{T}}\ \boldsymbol{e}\boldsymbol{H}_2^{\mathrm{T}}\ ...\ \boldsymbol{e}\boldsymbol{H}_\ell^{\mathrm{T}}) = (\boldsymbol{s}_1\ \boldsymbol{s}_2\ ...\ \boldsymbol{s}_\ell) \tag{4}$$

where $\boldsymbol{s}_l = \boldsymbol{e}\boldsymbol{H}_l^{\mathrm{T}}$ is the syndrome of length $b(n_0 - k_0)$, corresponding to the lth layer of $\boldsymbol{H}$, which can be written as

$$\boldsymbol{s}_l = (\boldsymbol{s}_{1,l}\ \boldsymbol{s}_{2,l}\ ...\ \boldsymbol{s}_{b,l})$$

where $\boldsymbol{s}_{j,l}$ is the syndrome of the jth constituent Hamming code in layer l, $j = 1, 2, ..., b$, $l = 1, 2, ..., \ell$. If at least one of the $n_0 - k_0$ parity-checks of that constituent code is not satisfied, then $\boldsymbol{s}_{j,l} \neq \boldsymbol{0}$, indicating that the constituent code is affected by one or more errors. When the syndrome $\boldsymbol{s}_{j,l}$ is nonzero, a maximum-likelihood decoder of that Hamming code assumes that a single error occurred at the position pointed at by the syndrome value.

For a given error pattern with W errors, we introduce the ℓ-tuple $\boldsymbol{u} = (u_1\ u_2\ ...\ u_\ell)$, where u_l, $l = 1, 2, ..., \ell$, denotes the number of constituent codes at the lth layer whose codewords are affected by a correctable error pattern (that is, by exactly one error). Note that $\boldsymbol{u}$ contains realizations of ℓ independent random variables that are integer-valued in the range $0 \le u_l \le b$, $l = 1, 2, ..., \ell$. Furthermore, let u denote the total number of constituent Hamming codes affected by exactly one error, that is,

$$u = |\boldsymbol{u}| = \sum_{l=1}^{\ell} u_l.$$

In other words, u is the number of constraint nodes in the Tanner graph that are connected to exactly one variable node with an erroneously received value. Clearly,

$$u \le W\ell \tag{5}$$

with equality if among the W variable nodes with erroneously received values,

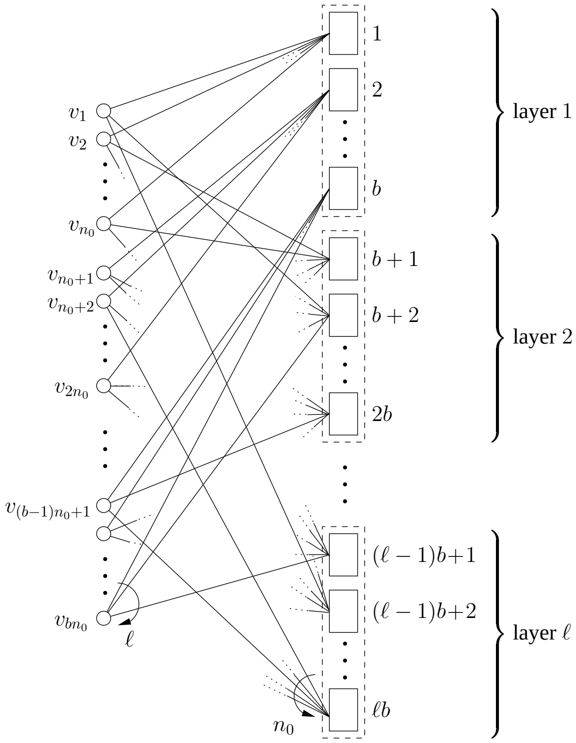

Figure 1: Tanner graph of an H-LDPC code defined by the parity-check matrix $\boldsymbol{H}$ given in (2). The graph illustrates the case when the first layer of $\boldsymbol{H}$ is the matrix $\boldsymbol{H}_\text{b}$ itself, *i.e.*, $\pi_1(\boldsymbol{H}_\text{b}) = \boldsymbol{H}_\text{b}$ (then the b constraint nodes in layer 1 are connected to the consecutive blocks of n_0 variable nodes). Other layers are obtained with arbitrary permutations.

there are no two connected to the same constraint node (that is, the W errors form a correctable error pattern for all affected constituent codes).

Figure 2 illustrates the values W, u, and $\boldsymbol{u}$ for an H-LDPC code with constituent $(7, 4, 3)$ Hamming codes and $\ell = 3$ layers. The error pattern illustrated in the figure has weight $W = 3$ and it affects 5 constituent codes, marked with black squares. Two of those five codes are affected by exactly one error, that is, $u = 2$ and $\boldsymbol{u} = (1\ 1\ 0)$.

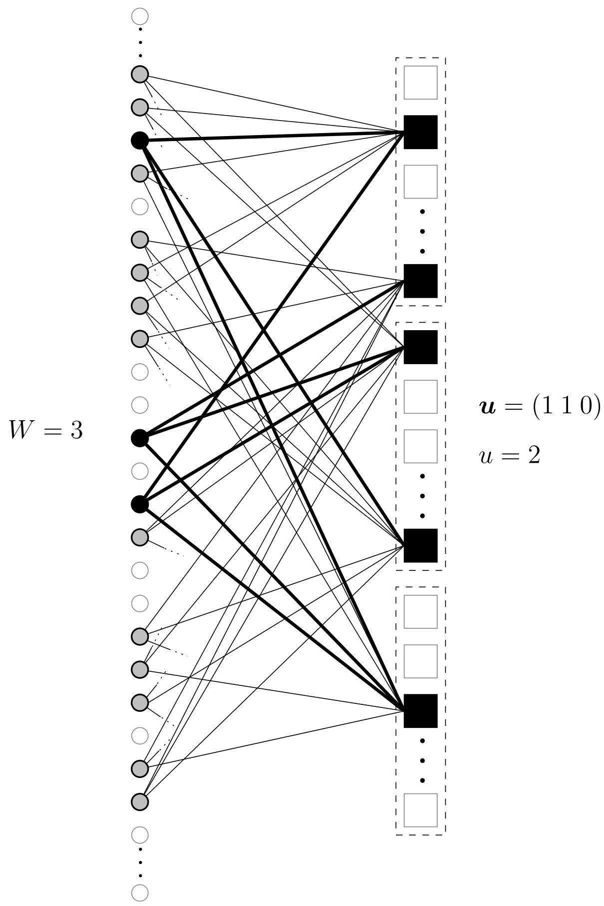

Figure 2: An illustration of the values u and W using the Tanner graph of an H-LDPC code with $\ell = 3$ layers and constituent code length $n_0 = 7$. $W = 3$ variable nodes with erroneously received values (black circles) are connected to 5 constraint nodes (black squares). These 5 constraint nodes are adjacent to 16 variable nodes, marked with gray and black circles. The number of constituent codes affected by exactly one error is $u = 2$ (one in layer 1 and one in layer 2).

3 Decoding Algorithm

Consider an iterative hard-decision decoding algorithm $\mathscr{A}$, whose decoding iterations i, $i = 1, 2, ..., i_{\max}$, consist of the following two steps:

(1) For the tentative sequence $\boldsymbol{r}^{(i)}$, where $\boldsymbol{r}^{(1)}$ is the received sequence $\boldsymbol{r}$, decode independently the ℓb constituent Hamming codes (that is, compute their syndromes $\boldsymbol{s}_{j,l}$, $j = 1, 2, ..., b$, $l = 1, 2, ..., \ell$, and if the value is nonzero, output the n_0-tuple where the position indicated by the syndrome is flipped). This yields ℓ independent decisions for each of the n symbols.

(2) Flip every symbol $r_k^{(i)}$, $k = 1, 2, ..., n$, in the sequence $\boldsymbol{r}^{(i)}$, for which *at least one* of the ℓ decisions requires that. This yields the updated sequence $\boldsymbol{r}^{(i+1)}$.

Assume that the error pattern $\boldsymbol{e}$ is such that the number of constituent codes affected by a single (correctable) error is larger than the number of codes affected by multiple (uncorrectable) errors. Then, during the first iteration of the algorithm $\mathscr{A}$, all correctable errors will be corrected, while the uncorrectable ones will result in erroneous decodings. Since Hamming codes are perfect single-error correcting codes with covering radius $\rho = 1$, each erroneous decoding will introduce at most one new error. Hence, the new error pattern, resulting from one decoding iteration has fewer errors than the initial error pattern. Clearly, if in each of the following iterations, the number of corrected errors is larger than the number of the newly introduced errors, then the total number of errors in $\boldsymbol{r}^{(i)}$ will decrease with the iteration number i and the algorithm yields the correct decision, *i.e.*, $\boldsymbol{r}^{(i_{\max})} = \boldsymbol{v}$. Then, we can state the following

Lemma 1 *For any H-LDPC code from the ensemble $\mathscr{C}(n_0, \ell, b)$, if an error pattern is such that in each decoding iteration of the algorithm $\mathscr{A}$ the number of corrected errors is larger than the number of introduced errors, then the algorithm $\mathscr{A}$ yields a correct decision after $\mathcal{O}(\log n)$ iterations, where $n = bn_0$ is the code length.*

Proof: Let $W = \omega n$ be the weight of the error pattern, and let ε denote a lower bound on the fraction of errors that are corrected in each iteration, $0 < \varepsilon < 1$. Then, after x iterations, the number of remaining errors is at most $\omega n(1-\varepsilon)^x$. The final decoding iteration $i_{\max}$ is reached when

$$\omega n(1-\varepsilon)^{i_{\max}} < 1$$

that is,

$$\log(\omega n) + i_{\max} \log(1-\varepsilon) < 0$$

which yields

$$i_{\max} < \frac{1}{\log\left(\frac{1}{1-\varepsilon}\right)} \log(\omega n). \tag{6}$$

Thus, the number of iterations is a logarithmic function of the code length. ■

The complexity of each decoding iteration of the algorithm $\mathscr{A}$ is proportional to the code length n. Thus, according to Lemma 1, the overall decoding complexity is $\mathcal{O}(n \log n)$, given that the number of correctable errors in the error sequence is larger than the number of the uncorrectable ones. The following lemma formulates a sufficient condition under which this holds.

Lemma 2 *For any error pattern with $w \leq W$ errors, if the number of constituent Hamming codes of an H-LDPC code from the ensemble $\mathscr{C}(n_0, \ell, b)$ that are affected by a single error is $u = \alpha w\ell$ with $\alpha > 1/3$, then it is guaranteed that the number of constituent codes with a single error is always larger than the number of constituent codes affected by uncorrectable errors.*

Proof: Consider the Tanner graph of an H-LDPC code, *cf.* Figure 2. The W variable nodes with erroneously received values are connected via $W\ell$ branches to $\leq W\ell$ constraint nodes. If $u > (1/3)W\ell$, then more than $(1/3)W\ell$ branches reach distinct constraint nodes. The remaining less than $(2/3)W\ell$ branches reach other constraint nodes, say, z, in such a way that at least two branches arrive to each of those z nodes. Hence, z is always smaller than $(1/3)W\ell$, that is, $z < u$, and we conclude that the number of codes affected by a single (correctable) error is larger than the number of codes affected by an uncorrectable error pattern. ■

Note that there is an important difference between the algorithm $\mathscr{A}$ and the decoding algorithm considered in [8]: in [8], a *majority rule* is applied for each symbol, that is, a symbol is flipped only if more than $\ell/2$ constituent SPC codes requires that. In the algorithm $\mathscr{A}$, however, a symbol is flipped as soon as *at least one* constituent Hamming code requires that.

4 Asymptotic Performance

As shown in the previous section, the iterative algorithm $\mathscr{A}$ corrects any error pattern with $W = \omega n$ or fewer errors, if the number of constituent codes affected by exactly one error is $u > (1/3)W\ell$. The question that arises, however, is whether such a code exists within the ensemble $\mathscr{C}(n_0, \ell, b)$. The following theorem allows us to receive the positive answer.

Theorem 1 *In the ensemble $\mathscr{C}(n_0, \ell, b)$ of H-LDPC codes, there exist codes (with probability p, where $\lim_{n\to\infty} p = 1$), which can correct any error pattern of* *weight up to $\omega_\alpha n$, with decoding complexity $\mathcal{O}(n \log n)$. The value ω_α is the largest root of the equation*

$$h(\omega) - \ell F(\alpha, \omega, n_0) = 0 \tag{7}$$

where $h(\omega) = -\omega \log_2 \omega - (1-\omega)\log_2(1-\omega)$ and the function $F(\alpha, \omega, n_0)$ is given by

$$F(\alpha, \omega, n_0) \triangleq h(\omega) - \frac{1}{n_0} h(\alpha\omega n_0) + \max\Bigg\{\omega \log_2 s - \frac{1}{n_0}\log_2\Big((1+s)^{n_0} - n_0 s\Big) + \alpha\omega \log_2\left(\frac{(1+s)^{n_0}}{n_0 s} - 1\right)\Bigg\} \tag{8}$$

where $\alpha > 1/3$ and the maximization is performed over all s such that

$$\frac{(1+s)^{n_0}}{n_0 s} \le \frac{1}{\alpha\omega n_0}.$$

Proof: For a *fixed* combination of $W = \omega n$ errors, the probability that the number of constituent Hamming codes of an H-LDPC code from the ensemble $\mathscr{C}(n_0, \ell, b)$ that are affected by a single error, will not exceed a certain value $\alpha W \ell$ is upper-bounded by:

$$P(u \le \alpha W \ell) \le 2^{-n\ell F(\alpha, \omega, n_0)} \tag{9}$$

where the function $F(\alpha, \omega, n_0)$ is given by (8). The proof of this statement follows Appendix 1 in [8] and is omitted here for brevity.

Now consider the probability that the number of constituent codes with a single error is at most $\alpha W \ell$ for *any* error pattern of a given weight W. If this probability is strictly smaller than 1, then there exist codes in the ensemble $\mathscr{C}(n_0, \ell, b)$ for which $u > \alpha W \ell$ for any weight-W error pattern. Thus, the existence of such codes is ensured if

$$\binom{n}{W} P(u \le \alpha W \ell) < 1.$$

Taking the logarithm and using the inequalities (9) and

$$\binom{n}{\omega n} \lesssim 2^{nh(\omega)}$$

where the asymptotic equality holds for $n \to \infty$, we readily obtain

$$h(\omega) - \ell F(\alpha, \omega, n_0) < 0. \tag{10}$$

The largest value of ω which satisfies (10) for a given α is ω_α. Finally, we have from Lemmas 1 and 2 that for $\alpha > 1/3$, the algorithm $\mathscr{A}$ corrects $\omega_\alpha n$ errors with complexity $\mathcal{O}(n \log n)$, which completes the proof. ■

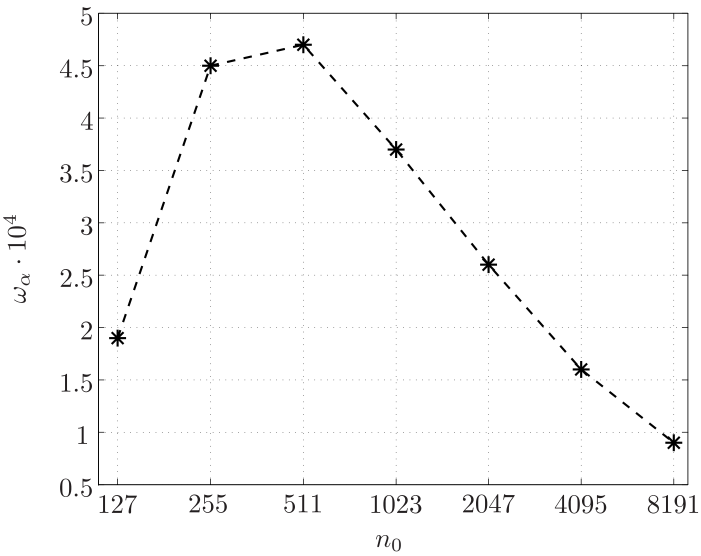

Figure 3: Values of ω_α computed for $\alpha = 0.334$ according to Theorem 1 for seven code ensembles of rates approximately $R \approx 1/2$, with the number of layers $\ell \in \{9, 16, 28, 51, 93, 171, 315\}$. The maximum is achieved with the constituent code length $n_0 = 511$ and $\ell = 28$.

Theorem 1 allows us to compute ω_α numerically for several choices of code parameters. The computations confirm the existence of codes with a nonvanishing ω_α. We use $\alpha = 0.334$, which is slightly above the limiting value 1/3. First, we consider code ensembles of rates close to 1/2. Figure 3 illustrates the values of ω_α computed for several such code ensembles $\mathscr{C}(n_0, \ell, b)$. With increasing n_0 (and, in order to keep the rate fixed, also with increasing ℓ) the value of ω_α increases only up to a certain point, $n_0 = 511$, where it reaches its maximum. With further increase of n_0 and ℓ, ω_α decays quickly.

Next we consider code ensembles of different rates, but with a fixed constituent code. Figure 4 illustrates the values ω_α for H-LDPC codes with the constituent $(511, 502, 3)$ Hamming code and with different code rates R, obtained by varying ℓ. We have found a nonvanishing ω_α for a wide range of rates, and its value decreases with increasing rate. Note that all the code ensembles considered in Figures 3 and 4 have minimum distances that almost meet the Varshamov-Gilbert bound, as shown in [14].

5 Conclusions

We have studied the performance of ensembles of Hamming code-based LDPC codes used over the BSC, when the code length n grows to infinity. It was shown

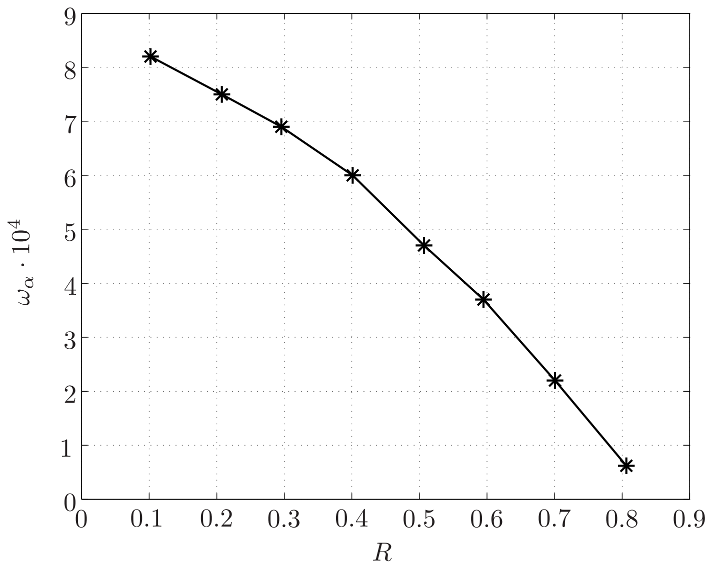

Figure 4: Values of ω_α computed for $\alpha = 0.334$ according to Theorem 1 for several code ensembles of different rates with the fixed constituent code length $n_0 = 511$ and with the number of layers $\ell \in \{11, 17, 23, 28, 34, 40, 45, 51\}$.

that these codes can be decoded with a simple iterative decoding algorithm whose complexity is $\mathcal{O}(n \log n)$, and that there exist H-LDPC codes which, when decoded with such an algorithm, are asymptotically capable of correcting a number of errors that grows linearly with the code length n. Such a property was previously proven to hold only for Gallager's LDPC codes and for the expander codes. The maximum fraction of errors correctable with the iterative decoder was computed numerically for two types of code ensembles, which are known to have minimum distances that asymptotically almost meet the Varshamov-Gilbert bound: codes of fixed rate $R \approx 1/2$ and codes of variable rates with a fixed constituent code.

References

[1] M. Tanner, A recursive approach to low complexity codes, *IEEE Trans. Inform. Theory* 27, 1981, 533-547.

[2] P. Elias, Error-free coding, *IEEE Trans. Inform. Theory* 4,1954, 29-37.

[3] R. G. Gallager, *Low-Density Parity-Check Codes*, MIT Press, Cambridge, MA, USA, 1963.

[4] M. Sipser and D. A. Spielman, Expander codes, *IEEE Trans. Inform. Theory* 42,1996, 1710-1722.

[5] G. Zémor, On expander codes, *IEEE Trans. Inform. Theory* 47, 2001, 835-837.

[6] I. E. Bocharova, B. D. Kudryashov, R. Johannesson, and V. V. Zyablov, Asymptotically good woven codes with fixed constituent convolutional codes, in *Proc. IEEE Int. Symp. Inf. Theory*, Nice, France, 2007.

[7] I. E. Bocharova, B. D. Kudryashov, R. Johannesson, and V. V. Zyablov, Woven graph codes over hyper graphs, in *Proc. 7th Int. ITG Conf. Source and Channel Coding*, Ulm, Germany, 2008.

[8] V. V. Zyablov and M. S. Pinsker, Estimation of the error-correction complexity for Gallager low-density codes, *Problems of Inform. Transmission* 11,1975, 23-26.

[9] A. Barg and G. Zémor, Error exponents of expander codes, *IEEE Trans. Inform. Theory* 48, 2002, 1725-1729.

[10] M. Lentmaier and K. Zigangirov, On generalized low-density parity-check codes based on Hamming component codes, *IEEE Commun. Letters* 3, 1999, 248-250.

[11] J. Boutros, O. Pothier, and G. Zémor, Generalized low density (Tanner) codes, in *Proc. IEEE Int. Comm. Conf. (ICC)*, Vancouver, Canada, 1, 1999, 441-445.

[12] N. Miladinović and M. Fossorier, Generalized LDPC codes with Reed-Solomon and BCH codes as component codes for binary channels, in *Proc. IEEE Global Comm. Conf. (GLOBECOM)*, St. Louis, MO, USA, 2005.

[13] M. Lentmaier and K. Zigangirov, Iterative decoding of generalized low-density parity-check codes, in *Proc. IEEE Int. Symp. Inf. Theory*, Cambridge, MA, USA, 1998.

[14] S. Stiglmayr and V. V. Zyablov, Asymptotically good low-density codes based on Hamming codes, in *Proc. XI Int. Symp. Problems of Redundancy in Inform. and Control Systems, available online at http://www.k36.org/redundancy2007/proceedings.php*, Saint Petersburg, Russia, 2007, 98-103.

[15] L. A. Bassalygo, "Asymptotic optimal switching systems," *Problems of Inform. Transmission* 17, 1981, 81-88.