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On the switching construction of Steiner quadruple systems [1]

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Abstract. The structure of Steiner quadruple system $S(v,4,3)$ of full 2-rank $v-1$ is considered. It is shown that there are two types (induced and singular) of such systems. It is shown that induced Steiner systems can be obtained from Steiner systems $S(v,4,3)$ of 2-rank $v-2$ by switching construction which is introduced here. Moreover, all non-isomorphic induced Steiner systems $S(16,4,3)$ of full 2-rank 15 are enumerated. It is found that there are 305616 such non-isomorphic systems $S(v,4,3)$, which are obtained from all 708103 non-isomorphic such systems of rank 14 studied earlier.

1 Introduction

A Stener system $S(n,k,t)$ is a pair (J,B) where J is a v-set and B is a collection of k-subsets of J such that every t-subset of J is contained in exactly one member of B. The necessary condition for existence of an SQS(v) is that $v \equiv 2$ or $4 \mod 6$. Hanani [1] proved that the necessary condition for the existence of an $S(v,4,3)$ is also sufficient. A Steiner system $S(v,4,3)$ is called resolvable if it can be split into mutually non-overlapping sets so that every set is a Steiner system $S(v,4,1)$. More on the Steiner systems can be found in [2-4] and on $S(16,4,3)$ in [5-8].

In this work, we consider the structure of the Steiner systems $S(v,4,3)$ of full 2-rank, i.e. of rank $v-1$ over $\mathbb{F}_2$. Any such system is one of two types, which we call *induced* and *singular*. The induced systems can be obtained by a switching operation from Steiner systems $S(v,4,3)$ of 2-rank $v-2$. This operation allows to construct Steiner systems of rank $r+1$ from systems of rank r. This operation, introduced later, is also interesting for the construction of resolvable Steiner systems. Namely, it keeps this property under certain conditions on the original resolvable systems.

The case $n=16$ is considered in details. In particular, we found exactly 305616 non-isomorphic induced Steiner systems $S(16,4,3)$, which were constructed by the switching operation from all 708103 non-isomorphic systems $S(16,4,3)$ of rank 14. We described the structure of singular Steiner systems.

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2 Preliminary results

Let $E = \{0, 1\}$. A binary code of length n is an arbitrary subset of E^n. Denote a binary code C with length n, with minimum distance d and cardinality N as a (n, d, N)-code. Denote by $\mathrm{wt}(\boldsymbol{x})$ the Hamming weight of vector $\boldsymbol{x}$ over E. For a (binary) code C denote by $\langle C \rangle$ the linear envelope of words of C over $\mathbb{F}_2$. The dimension of space $\langle C \rangle$ is called the *rank* of C over $\mathbb{F}_2$ and is denoted $\mathrm{rank}(C)$.

Denote by (n, w, d, N) a binary constant weight code C of length n, with weight of all codewords w, with minimum distance d and cardinality N. For vector $\boldsymbol{v} = (v_1, ..., v_n) \in E^n$ denote by $\mathrm{supp}(\boldsymbol{v})$ its support: $\mathrm{supp}(\boldsymbol{v}) = \{i : v_i \neq 0\}$.

The binary (n, d, N)-code A which is a linear k-dimensional space over $\mathbb{F}_2$ is denoted by $[n, k, d]$-code. For binary vector $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n)$ denote by $(\boldsymbol{x} \cdot \boldsymbol{y}) = x_1 y_1 + \cdots + x_n y_n$ their inner product over $\mathbb{F}_2$. For any (n, d, N)-code (linear, nonlinear, or constant weight) denote by $C^\perp$ its dual code: $C^\perp = \{\boldsymbol{v} \in \mathbb{F}_2^n : (\boldsymbol{v} \cdot \boldsymbol{c}) = 0, \ \forall \ \boldsymbol{c} \in C\}$. Clearly $C^\perp$ is a linear $[n, n-k, d^\perp]$-code with some minimum distance $d^\perp$, where $k = \mathrm{rank}(C)$.

Denote by E_2^n the set of all binary vectors of length n of weight 2. Let $J_n = \{1, 2, \ldots, n\}$ be the coordinate set of E^n and let $\mathcal{S}_n$ be the full group of permutations of n elements (thus $|\mathcal{S}_n| = n!$). A binary incidence matrix of a Steiner system $S(v, 4, 3)$ is a constant weight $(v, 4, 4, v(v-1)(v-2)/24)$-code C which is strongly optimal [8]. In our notation the connection between the system (X, B) and the code C is: $B = \{\mathrm{supp}(\boldsymbol{v}) \subset X : \boldsymbol{v} \in C\}$. In this note, the Steiner system $S(v, 4, 3)$ is identified with the constant weight $(v, 4, 4, v(v-1)(v-2)/24)$-code, which uniquely defines this system [8].

Definition 1 *Two Steiner systems (X, B) and (X', B') of order n are isomorphic, if their incidence matrices S and S' are equivalent as constant weight codes, i.e. if there exists some permutation $\tau \in \mathcal{S}_n$ such that S and $\tau S'$ coincide up to the permutation of rows.*

3 Switching constructions of SQS(v)

Let C be a Steiner system $S(v, 4, 3)$ of rank $r \leq v - 2$ over $\mathbb{F}_2$. Applying the appropriate permutation of coordinates, C can be presented in the form, when the $[v, v/2, 2]$-code $C^\perp$, orthogonal to $\langle C \rangle$, is of the following form:

$$C^\perp = \{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_1 + \boldsymbol{u}_2\}, \tag{1}$$

where $\boldsymbol{u}_0$ is the zero vector, $\boldsymbol{u}_1 = (11\ldots1|00\ldots0)$, and $\boldsymbol{u}_2 = (00\ldots0|11\ldots1)$. Thus we split n coordinates into two blocks of $v/2$ coordinates such that any $\boldsymbol{c} \in C$ consists of two vectors $\boldsymbol{c} = (\boldsymbol{c}_1 \,|\, \boldsymbol{c}_2)$ where each vector $\boldsymbol{c}_i$ satisfies to the overall parity checking: $\mathrm{wt}(\boldsymbol{c}_i) \equiv 0 \pmod 2$, $i = 1, 2$ (we call it a *parity rule*).

Definition 2 *Let C be a Steiner system $S(v,4,3)$ of rank less or equal to $v-2$ over $\mathbb{F}_2$ with orthogonal code* (1)*. Define the subset $C_{(w_1|w_2)}$ of C where $w_1, w_2 \in \{0,2,4\}$ as follows:*

$$C_{(w_1|w_2)} \;=\; \{\boldsymbol{c} = (\boldsymbol{a}\,|\,\boldsymbol{b}) \in C :\; \mathrm{wt}(\boldsymbol{a}) = w_1,\; \mathrm{wt}(\boldsymbol{b}) = w_2\}.$$

Lemma 1 *Let $v \geq 16$ be an integer such that $v/2 \equiv 2$ or 4 (mod 6) and let C be a Steiner system $S(v,4,3)$ of rank less or equal to $v-2$ over $\mathbb{F}_2$ with dual code* (1)*. Then C is a union of three subsets*

$$C \;=\; C_{(4|0)} \bigcup C_{(0|4)} \bigcup C_{(2|2)}$$

where $C_{(4|0)}$ (respectively $C_{(0|4)}$) is a Steiner system $S(v/2,4,3)$ and $C_{(2|2)}$ has cardinality $\binom{v/2}{2} \times (v/2-1)$.

Definition 3 *Define the following (constant weight) $(8,4,4,8)$-codes:*

$$C_P = \left\{ \begin{array}{cc} (1111|0000), & (0000|1111), \\ (1100|1100), & (0011|0011), \\ (1010|1010), & (0101|0101), \\ (1001|0110), & (0110|1001) \end{array} \right\},\; C_N = \left\{ \begin{array}{cc} (1110|1000), & (1101|0100), \\ (1011|0010), & (0111|0001), \\ (1000|1110), & (0100|1101), \\ (0010|1011), & (0001|0111) \end{array} \right\}.$$

For a given permutation $\pi \in \mathcal{S}_4$ denote by $C_{\pi(P)}$ (respectively, by $C_{\pi(N)}$) the code obtained from C_P (respectively, from C_N) by applying π to the last 4 columns of the code C_P (respectively C_N).

Note that the middle six columns of C_P define two Pasch configurations.

Theorem 1 (*switching construction*). *Let S be a Steiner system $S(v,4,3)$ and let C be the corresponding constant weight $(v,4,4,v(v-1)(v-2)/24)$-code with dual code* (1)*. Assume that C contains as a subcode the code $C_{\pi(P)}$ for some $\pi \in \mathcal{S}$. Define the new code*

$$C^*(\pi(P)) \;=\; (C \setminus C_{\pi(P)}) \cup C_{\pi(N)}.$$

Then:
1). *The set $C^* = C^*(\pi(P))$ is a constant weight $(v,4,4,v(v-1)(v-2)/24)$-code, which defines a new Steiner system $S(v,4,3)$, denoted by $S^* = S^*(\pi(P))$.*
2). *The new system S^* is not isomorphic to the initial system S (since they have different number of Pasch configurations).*
3). *If the initial system S is resolvable and if the code $C_{\pi(P)}$ belongs to exactly four parallel classes of C, then the resulting system S^* is resolvable too.*

4 The structure of Steiner systems $S(v,4,3)$ with rank $v-1$ over $\mathbb{F}_2$

Let $S = S(v,4,3)$ be of rank $v-1$ over $\mathbb{F}_2$. Recall $J = \{1,2,\dots,v\}$ is the coordinate set of S. Divide J into two arbitrary equal halves: J_1 and J_2. Applying some permutation $\pi \in \mathcal{S}_n$, any vector $\boldsymbol{c} \in \pi(C)$ can be presented in the form $\boldsymbol{c} = (\boldsymbol{c}_1 \,|\, \boldsymbol{c}_2)$, where $\mathrm{supp}(\boldsymbol{c}_i) \in J_i$ for $i = 1,2$. Hence without loss of generality assume that J_1 is the left half of J and J_2 is the right half of J_2.

Definition 4 *For any Steiner system $S(v,4,3)$ of rank $v-1$ over $\mathbb{F}_2$ define the left and right spectrum (x_i, y_i, z_i), $i = 1,2$ as follows:*

$$\begin{aligned} x_i &= |\{\boldsymbol{c} = (\boldsymbol{c}_1 \,|\, \boldsymbol{c}_2):\ \mathrm{wt}(\boldsymbol{c}_i) = 4\}|\,, \\ y_i &= |\{\boldsymbol{c} = (\boldsymbol{c}_1 \,|\, \boldsymbol{c}_2):\ \mathrm{wt}(\boldsymbol{c}_i) = 3\}|\,, \\ z_i &= |\{\boldsymbol{c} = (\boldsymbol{c}_1 \,|\, \boldsymbol{c}_2):\ \mathrm{wt}(\boldsymbol{c}_i) = 2\}|\,. \end{aligned}$$

Lemma 2 *Let C be an arbitrary Steiner system $(v,4,3)$ of rank $v-1$ over $\mathbb{F}_2$. Then $x = x_1 = x_2$, $y = y_1 = y_2$, $z = z_1 = z_2$. Furthermore*

$$y \;=\; \binom{v/2}{3} \;-\; 4\,x, \quad z \;=\; 6\,x \;+\; \binom{v/2}{2}. \tag{2}$$

Clearly for the same system the numbers x, y and z depend on the choice of subsets J_i.

Definition 5 *For a Steiner system $S = S(v,4,3)$ of rank $v-1$ over $\mathbb{F}_2$ define the spectrum (x,y,z), where x takes the maximal value for given S and y and z satisfies* (2).

Lemma 3 *For a Steiner system $S(v,4,3)$ of rank $v-1$ over $\mathbb{F}_2$ with spectrum (x,y,z), we have*

$$x \;\geq\; \left\lceil \frac{v(v-1)(v-2)}{24} \cdot \frac{\binom{v-4}{v/2}}{\binom{v}{v/2}} \right\rceil .$$

In particularly, $x \geq 6$ when $v = 16$.

Definition 6 *We say that* 4 *different binary vectors of length v and weight* 3 *form a* 4*-clique, if*

$$|\cup_{i=1}^{4} \mathrm{supp}(\boldsymbol{y}_i)| \;=\; 4.$$

Lemma 4 *Let X be a constant weight $(v,4,4,x)$ code with cardinality $x \leq v(v-1)(v-2)/24-2$. Denote by Y the constant weight $(v,3,2,y)$ code, formed by all vectors of weight* 3*, which are not covered by codewords of X, i.e. $y = \binom{v}{3} - 4\,x$. Then X can be imbedded into a Steiner system $S(v,4,3)$, if and only if all the codewords of Y can be partitioned into disjoint* 4*-cliques $C_1, \dots, C_k$, $k = y/4$, such that $|\mathrm{supp}(C_i) \cap \mathrm{supp}(C_j)| \leq 2$ for any $i \neq j$.*

5 Induced Steiner systems $S(v, 4, 3)$

We say that a Steiner system $S = S(v, 4, 3)$ of full rank $r = v - 1$ is *induced*, if it is obtained by the switching construction from some Steiner system $S' = S(v, 4, 3)$ of rank $\leq v - 2$. In the contrary case, we call this system *singular*.

Theorem 2 *Let $S = S(v, 4, 3)$ be a Steiner system of rank $r = v - 1$ over $\mathbb{F}_2$ with spectrum (x, y, z) and let v is a multiple of* 4. *Let X_i and Y_j be the corresponding $(v/2, 4, 4, x)$- and $(v/2, 3, 2, y)$-codes, where y satisfies* (2) *and $i, j \in \{1, 2\}$. If X_1 and X_2 are any subcodes of a Steiner system $S' = S(v/2, 4, 3)$, then S is an induced system.*

It is known from [6,7] there are exactly 708103 non-isomorphic Steiner systems SQS(16) of rank 14 over $\mathbb{F}_2$. By computations it was found that all these 708103 systems give 295488 different Pasch configurations. For each system SQS(16) of rank 14, containing some Pasch configurations we have applied all possible switchings.

Theorem 3 (*Computational results*). *There are* 305616 *non-isomorphic induced Steiner systems $S(16, 4, 3)$ of rank* 15 *over $\mathbb{F}_2$. They are obtained from* 708103 *non-isomorphic Steiner systems SQS*(16) *of rank* 14 *over $\mathbb{F}_2$ by applying all possible switchings.*

Remark 1 *Taking into account the result of* [7] *we conclude that there are exactly* 27715 *non-isomorphic singular Steiner systems $S(16, 4, 3)$ of rank* 15.

6 Derived triple systems

For a system $S(v, 4, 3)$, given by the pair of sets (J, B), a derived triple system of (J, B) is a pair (J_a, B_a), where $J_a = J \backslash \{a\}$ and $B_a = \{b \backslash \{a\} : a \in b \in B\}$. It is obvious, that every derived triple system is a Steiner triple system $S(v-1, 3, 2)$. For $v = 16$ we obtain a system $S(15, 3, 2)$. It is known [9] that there are exactly 80 non-isomorphic systems $S(15, 3, 2)$. There is a standard numbering of these systems by the indices from 1 to 80, related to the number of Pasch configurations (see [4]).

Given a system $S = S(v, 4, 3)$, let $\beta = \beta(S)$ denote the number of its pairwise non-isomorphic $S(v - 1, 3, 2)$. Clearly $1 \leq \beta \leq v$ for any S. A system S is said to be *homogeneous* (respectively, *heterogeneous*), if $\beta = 1$ (respectively, $\beta = v$). Among all induced Steiner systems $S(16, 4, 3)$, the derived systems $S(15, 3, 2)$ that we found are those with indices $1, 2, \ldots, 77$ missing $35, 38, 43, 68, 69, 70, 73, 74$, i.e. all together 69 non-isomorphic $S(15, 3, 2)$ out of total 80 such systems. All Steiner triple systems with these numbers occur as derived in the homogeneous $S(16, 4, 3)$.

Denote by $N_{hom}(i)$ the number of non-isomorphic homogeneous systems $S(16, 4, 3)$ with rank 15, whose derived systems are $S(15, 3, 2)$ with number i, where $i \in \{1, 2, \ldots, 7\}$. Denote by $N(\beta)$ the number of such non-isomorphic systems $S(16, 4, 3)$ with rank 15 with given β. Denote by $N(\mu(i_1), \mu(i_2), \ldots, \mu(i_\beta))$ the number of non-isomorphic systems $S(16, 4, 3)$ with rank 15 which have $\mu(i_s) > 0$ derived systems with index i_s, where $i_s \in \{1, 2, \ldots, 7\}$ for $s = 1, \ldots, \beta$, i.e. in our notation $N_{hom}(i) = N(\mu(i) = 16)$.

Proposition 1 (*Computational results*). *Among the non-isomorphic induced* $S(16, 4, 3)$ *of rank* 15 *over* $\mathbb{F}_2$*, there are* 245 *homogeneous systems. Among these systems there are*:

$$\begin{array}{rclrcl} N_{hom}(4) &=& 1, & N_{hom}(8) &=& 12, \\ N_{hom}(9) &=& 3, & N_{hom}(10) &=& 6, \\ N_{hom}(11) &=& 1, & N_{hom}(12) &=& 15, \\ N_{hom}(13) &=& 6, & N_{hom}(14) &=& 2, \\ N_{hom}(16) &=& 1, & N_{hom}(17) &=& 1, \\ N_{hom}(20) &=& 2, & N_{hom}(23) &=& 6, \\ N_{hom}(24) &=& 5, & N_{hom}(25) &=& 52, \\ N_{hom}(26) &=& 64, & N_{hom}(27) &=& 5, \\ N_{hom}(28) &=& 5, & N_{hom}(29) &=& 14, \\ N_{hom}(30) &=& 1, & N_{hom}(32) &=& 10, \\ N_{hom}(33) &=& 5, & N_{hom}(34) &=& 5, \\ N_{hom}(36) &=& 2, & N_{hom}(52) &=& 1, \\ N_{hom}(53) &=& 3, & N_{hom}(54) &=& 5, \\ N_{hom}(59) &=& 2, & N_{hom}(60) &=& 3, \\ N_{hom}(63) &=& 2, & N_{hom}(64) &=& 2, \\ N_{hom}(65) &=& 2, & N_{hom}(71) &=& 1, \end{array}$$

Proposition 2 (*Computational results*). *For induced Steiner systems* $S(16, 4, 3)$ *of rank* 15 *over* $\mathbb{F}_2$*, the distribution of the value* $N(\beta)$ *is the following*:

$$\begin{array}{rclrcl} N(1) &=& 245, & N(2) &=& 1412, \\ N(3) &=& 2732, & N(4) &=& 7553, \\ N(5) &=& 9674, & N(6) &=& 19187, \\ N(7) &=& 19187, & N(8) &=& 33896, \\ N(9) &=& 47645, & N(10) &=& 57794, \\ N(11) &=& 57794, & N(12) &=& 34250, \\ N(13) &=& 15607, & N(14) &=& 4758, \\ N(15) &=& 884, & N(16) &=& 77. \end{array}$$

Taking into account the results of [7], among all Steiner systems $S(16, 4, 3)$ there are 77 heterogeneous induced systems of rank 15.

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