On the \((2, 1)\)-extendability of ternary linear codes

Yuri Yoshida
Tatsuya Maruta
maruta@mi.s.osakafu-u.ac.jp
Department of Mathematics and Information Sciences, Osaka Prefecture University, Sakai, Osaka 599-8531, JAPAN

Abstract. We show that every \([n, k, d]_3\) code with diversity \((\Phi_0, \Phi_1)\), \(3 \leq k \leq 5\), \(\gcd(d, 3) = 1\), is \((2, 1)\)-extendable except for the case \((\Phi_0, \Phi_1) = (40, 36)\) for \(k = 5\), and that an \([n, 5, d]_3\) code with diversity \((40, 36)\), \(\gcd(d, 3) = 1\), is \((2, 1)\)-extendable if \(A_d \leq 50\). Geometric conditions for the \((2, 1)\)-extendability of not necessarily extendable \([n, k, d]_3\) codes for \(k = 5, 6\) are also given.

1 Introduction

Let \(F_q^n\) denote the vector space of \(n\)-tuples over \(F_q\), the field of \(q\) elements. A linear code \(C\) of length \(n\), dimension \(k\) and minimum (Hamming) distance \(d\) over \(F_q\) is referred to as an \([n, k, d]_q\) code. The weight of a vector \(x \in F_q^n\), denoted by \(\text{wt}(x)\), is the number of nonzero coordinate positions in \(x\). The weight distribution of \(C\) is the list of numbers \(A_i\) which is the number of codewords of \(C\) with weight \(i\). The weight distribution with \((A_0, A_d, \ldots) = (1, \alpha, \ldots)\) is also expressed as \(0^1d^\alpha\ldots\). We only consider non-degenerate codes having no coordinate which is identically zero.

For an \([n, k, d]_q\) code \(C\) with a generator matrix \(G\), \(C\) is called \((l, s)\)-extendable (to \(C'\)) if there exist \(l\) vectors \(h_1, \ldots, h_l \in F_q^k\) so that the extended matrix \([G, h_1^T, \ldots, h_l^T]\) generates an \([n + l, k, d + s]_q\) code \(C'\) ([1]). Then \(C'\) is called an \((l, s)\)-extension of \(C\). \'(1, 1)\)-extendable' is simply called extendable. In this paper we are concerned with \((2, 1)\)-extendability of ternary linear codes with dimension \(k \leq 6\).

Let \(C\) be an \([n, k, d]_3\) code with \(k \geq 3\), \(\gcd(3, d) = 1\). The diversity \((\Phi_0, \Phi_1)\) of \(C\) is given as the pair of integers:

\[
\Phi_0 = \frac{1}{2} \sum_{3 \mid i, i \neq 0} A_i, \quad \Phi_1 = \frac{1}{2} \sum_{i \neq 0, d \ (\text{mod} \ 3)} A_i,
\]

where the notation \(x \mid y\) means that \(x\) is a divisor of \(y\).
Let \( D_k \) be the set of all possible diversities of \( C \). \( D_k \) has been determined in [3] for \( k \leq 6 \) and in [5] for \( k \geq 7 \). For \( k \geq 3 \), let \( D_k^+ \) and \( D_k^- \) be as follows:

\[
D_k^- = \{(\theta_{k-2}, 0), (\theta_{k-3}, 2 \cdot 3^{k-2}), (\theta_{k-2}, 2 \cdot 3^{k-2}), (\theta_{k-2} + 3^{k-2}, 3^{k-2})\}, \quad D_k^+ = D_k \setminus D_k^-,
\]

where \( \theta_j = (3^{j+1} - 1)/2 \). It is known that \( D_k^\pm \) is included in \( D_k \) and that \( C \) is extendable if \((\Phi_0, \Phi_1) \in D_k^\pm \) ([3]). The necessary and sufficient conditions for the extendability of \( C \) with \((\Phi_0, \Phi_1) \in D_k^\pm \) are given in [3-7] for \( k \leq 6 \).

We denote by \( \text{PG}(r, q) \) the projective geometry of dimension \( r \) over \( \mathbb{F}_q \). A \( j \)-flat is a projective subspace of dimension \( j \) in \( \text{PG}(r, q) \). 0-flats, 1-flats, 2-flats, 3-flats, \((r - 2)\)-flats and \((r - 1)\)-flats are called points, lines, planes, solids, secundums and hyperplanes, respectively.

Let \( C \) be an \([n, k, d]_q \) code with a generator matrix \( G = [g_1, \ldots, g_k]^T \). For \( P = P(p_1, \ldots, p_k) \in \Sigma \), the weight of \( P \) with respect to \( C \) is defined in [4] as

\[
w_C(P) = wt(\sum_{i=1}^k p_ig_i).
\]

From now on, let \( C \) be an \([n, k, d]_3 \) code with \( \text{gcd}(d, 3) = 1 \), \( k \geq 3 \). Let

\[
\begin{align*}
F_0 &= \{P \in \Sigma \mid w_C(P) \equiv 0 \pmod{3}\}, \\
F_2 &= \{P \in \Sigma \mid w_C(P) \equiv d \pmod{3}\}, \\
F_d &= \{P \in \Sigma \mid w_C(P) = d\}, \\
F_1 &= \Sigma \setminus (F_0 \cup F_2), \\
F &= F_0 \cup F_1.
\end{align*}
\]

**Lemma 1.1** ([4]). \( C \) is \((2, 1)\)-extendable iff there exist two hyperplanes \( H_1, H_2 \) of \( \Sigma \) such that \( F_d \cap H_1 \cap H_2 = \emptyset \). Equivalently, \( F \cup F_e \) contains a secundum of \( \Sigma \).

We give the necessary and sufficient conditions for the \((2, 1)\)-extendability of \([n, k, d]_3 \) codes with diversity \((\Phi_0, \Phi_1) \), \( 3 \leq k \leq 6 \), \( d \equiv 1 \) or 2 \((\text{mod} \ 3)\) from this geometrical point of view.

A \( t \)-flat \( \Pi \) of \( \Sigma \) with \( |\Pi \cap F_0| = i \), \( |\Pi \cap F_1| = j \) is called an \((i,j)_t \) flat. An \((i,j)_1 \) flat is called an \((i,j)_t \) line. An \((i,j)_t \)-plane and an \((i,j)_t \)-solid are defined similarly. We denote by \( F_j \) the set of \( j \)-flats of \( \Sigma \). Let \( \Lambda_t \) be the set of all possible \((i,j)_t \) for which an \((i,j)_t \) flat exists in \( \Sigma \). Then we have

\[
\begin{align*}
\Lambda_1 &= \{(1, 0), (0, 2), (2, 1), (1, 3), (4, 0), \}\), \\
\Lambda_2 &= \{(4, 0), (1, 6), (4, 3), (4, 6), (7, 3), (4, 9), (13, 0), \}\), \\
\Lambda_3 &= \{(13, 0), (4, 18), (13, 9), (10, 15), (16, 12), (13, 18), (22, 9), (13, 27), (40, 0), \}\), \\
\Lambda_4 &= \{(40, 0), (13, 54), (40, 27), (31, 45), (40, 36), (40, 45), (49, 36), (40, 54), (67, 27), \}\).
\]
(40, 81), (121, 0)).

\(\Lambda_5 = \{(121, 0), (40, 162), (121, 81), (94, 135), (121, 108), (112, 126), (130, 117),
(121, 135), (148, 108), (121, 162), (202, 81), (121, 243), (364, 0)\},

see [3]. Let \(\Pi_t \in \mathcal{F}_t\). Denote by \(c_i^{(t)}\) the number of \((i, j)_{t-1}\) flats in \(\Pi_t\) and let \(\varphi_s^{(t)} = |\Pi_t \cap F_s|, \ s = 0, 1\). \((\varphi_s^{(0)}, \varphi_1^{(1)})\) is called the diversity of \(\Pi_t\) and the list of \(c_i^{(t)}\)’s is called its spectrum. Thus \(\Lambda_t\) is the set of all possible diversities of \(\Pi_t\).

According to the diversity of \(\mathcal{C}\) we give the necessary and sufficient conditions for the \((2,1)\)-extendability of \(\mathcal{C}\) using some of the following six conditions:

For \(k \geq 4\), let \((C_{k-0}), (C_{k-1})\) and \((C_{k-2})\) be the following conditions:

\((\text{C}_{k-0})\) (\(C_{k-0}\)) holds and there exists \((\theta_{k-4}, 0)_{k-3}\) flat \(\delta_1\) in \(\Sigma\) satisfying \(\delta_1 \setminus F_0 \subset F_e\);

\((\text{C}_{k-1})\) there exist two \((\theta_{k-4}, 0)_{k-3}\) flats \(\delta_1, \delta_2\) in \(\Sigma\) such that \(\delta_1 \cap \delta_2\) is a \((\theta_{k-4}, 0)_{k-4}\) flat;

\((\text{C}_{k-2})\) there exist two \((\theta_{k-4}, 0)_{k-3}\) flats \(\delta_1, \delta_2\) in \(\Sigma\) such that \(\delta_1 \cap \delta_2\) is a \((\theta_{k-4}, 0)_{k-4}\) flat with \(\delta_1 \cup \delta_2 \setminus (\delta_1 \cap \delta_2) \subset F_e\);

For \(k \geq 4\), let \((C_{k-3})\) and \((C_{k-4})\) be the following conditions:

\((\text{C}_{k-3})\) there are three non-collinear points \(Q_1, Q_2, Q_3 \in F_e\) such that the three lines \(\langle Q_1, Q_2 \rangle, \langle Q_2, Q_3 \rangle, \langle Q_3, Q_1 \rangle\) are \((0, 2)\)-lines;

\((\text{C}_{k-4})\) there exist three \((\theta_{k-5}, 0)_{k-4}\) flats \(\delta_1, \delta_2, \delta_3\) through a fixed \((\theta_{k-5}, 0)_{k-5}\) flat \(L\) such that \(\delta_1, \delta_2, \delta_3, \delta_1\) form distinct \((\theta_{k-5}, 2 \cdot 3^{k-4})_{k-3}\) flats and that \((\delta_1 \cup \delta_2 \cup \delta_3) \setminus L \subset F_e\) holds;

\((\text{C}_{k-3})\) there are three non-collinear points \(P_1, P_2, P_3 \in F_1\) such that the three lines \(\langle P_1, P_2 \rangle, \langle P_2, P_3 \rangle, \langle P_3, P_1 \rangle\) are \((0, 2)\)-lines each of which contains two points of \(F_e\);

\((\text{C}_{k-4})\) there exist a \((\theta_{k-5}, 0)_{k-5}\) flat \(L\), three \((\theta_{k-5}, 3^{k-4})_{k-4}\) flats \(\delta_1', \delta_2', \delta_3'\) through \(L\), and six \((\theta_{k-5}, 0)_{k-4}\) flats \(\delta_1, \cdots, \delta_6\) through \(L\) such that \(\langle \delta_i', \delta_j' \rangle\) forms a \((\theta_{k-5}, 2 \cdot 3^{k-4})_{k-3}\) flat containing two of \(\delta_1, \cdots, \delta_6\) for \(1 \leq i < j \leq 3\) and that \((\cup_{i=1}^{6} \delta_i) \setminus L \subset F_e\) holds.

For \(k = 5\), let \((C_{k-5})\) and \((C_{k-6})\) be the following conditions:

\((\text{C}_{k-5})\) there exist a \((4, 0)\)-line \(l\) and four skew \((1, 0)\)-lines \(l_1, l_2, l_3, l_4\) such that each of \(l_1, \ldots, l_4\) meets \(l\) and that \(\langle l_1, l_2, l_3, l_4 \rangle \in \mathcal{F}_3\) and \((\cup_{i=1}^{4} l_i) \setminus l \subset F_e\) hold;

\((\text{C}_{k-6})\) there exist a \((4, 3)\)-plane \(\delta\) in \(\Sigma\) and a point \(R \in F_e\) such that \(l_i = \langle R, P_i \rangle\) is a \((1, 0)\)-line for \(i = 1, 2, 3, 4\) and \((\delta \cup l_1 \cup l_2 \cup l_3 \cup l_4) \cap F_d = \emptyset\), where \(F_0 \cap \delta = \{P_1, \ldots, P_4\}\).

**Theorem 1.2.** Let \(\mathcal{C}\) be an \([n, k, d]_3\) code with diversity \((\Phi_0, \Phi_1)\), \(k = 3\) or \(4\), \(\gcd(3, d) = 1\). Then \(\mathcal{C}\) is \((2, 1)\)-extendable.

**Theorem 1.3.** Let \(\mathcal{C}\) be an \([n, 5, d]_3\) code with diversity \((\Phi_0, \Phi_1) \neq (40, 36), \gcd(3, d) = 1\). Then \(\mathcal{C}\) is \((2, 1)\)-extendable.
Theorem 1.4. Let \( C \) be an \([n, 5, d]_3\) code with diversity \((40, 36)\), \(\gcd(3, d) = 1\). Then \( C \) is \((2, 1)\)-extendable iff one of the conditions \((C_{4-0})\), \((C_{4-3})\) holds.

Theorem 1.5. Let \( C \) be an \([n, 5, d]_3\) code with diversity \((40, 36)\), \(\gcd(3, d) = 1\). Then \( C \) is \((2, 1)\)-extendable if \( A_d \leq 50 \).

Theorem 1.6. Let \( C \) be an \([n, 6, d]_3\) code with diversity \((\Phi_0, \Phi_1) \notin \{(121, 108), (112, 126), (130, 117)\}\), \(\gcd(3, d) = 1\). Then \( C \) is \((2, 1)\)-extendable.

Theorem 1.7. Let \( C \) be an \([n, 6, d]_3\) code with diversity \((121, 108)\), \(\gcd(3, d) = 1\). Then \( C \) is \((2, 1)\)-extendable iff one of the conditions \((C_{5-1}), (C_{5-3}), (C_{5-4}), (C_{5-5}), (C_{5-6})\) holds.

Theorem 1.8. Let \( C \) be an \([n, 6, d]_3\) code with diversity \((112, 126)\), \(\gcd(3, d) = 1\). Then \( C \) is \((2, 1)\)-extendable iff one of the conditions \((C_{5-2}), (C_{5-3}), (C_{5-4}), (C_{5-5}), (C_{5-6})\) holds.

Theorem 1.9. Let \( C \) be an \([n, 6, d]_3\) code with diversity \((130, 117)\), \(\gcd(3, d) = 1\). Then \( C \) is \((2, 1)\)-extendable iff one of the conditions \((C_{5-0}), (C_{5-3}), (C_{5-4}), (C_{5-5}), (C_{5-6})\) holds.

Example. Let \( C \) be a \([15, 5, 8]_3\) code with a generator matrix

\[
G = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & 2 & 1 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 1
\end{bmatrix},
\]

whose weight distribution is \(0^1 8^{60} 9^{40} 10^{62} 11^{20} 12^{40} 13^{10} 14^{10}\) (diversity \((40, 36)\)). Then we can take \( Q_1 = (0, 0, 1, 1, 1), Q_2 = (1, 2, 1, 2, 2), Q_3 = (1, 1, 0, 0, 1) \) so that the condition \((C_{4-3})\) of Theorem 1.4 holds. Since \( V(\overline{x_0} + 2\overline{x_1} + 2\overline{x_2} + x_3) \cap V(\overline{x_0} + \overline{x_2} + 2\overline{x_4}) = \langle Q_1, Q_2, Q_3 \rangle \), by adding the column \((1, 2, 2, 1, 0)^T\) and \((1, 0, 1, 0, 2)^T\) to \( G \), we get a \((2,1)\)-extension of \( C \) whose weight distribution is \(0^{1} 9^{48} 10^{56} 11^{46} 12^{34} 13^{30} 14^{26} 15^{8} 16^{4}\).

2. Proof of Theorems 1.2–1.4, 1.6–1.9.

Proof of Theorem 1.2. When \( k = 3 \), there is a point \( P \notin F_d \) iff \( C \) is \((2,1)\)-extendable. Obviously, any plane have such a point \( P \) in \( F \). When \( k = 4 \), there is an \((i, j)\)-line \( L \) with \( L \cap F_d = \emptyset \) iff \( C \) is \((2,1)\)-extendable. From Table 1 and Table 2 in [3], it can be checked that any solid has an \((i, j)\)-line with \((i, j) = (1, 3) \) or \((4, 0)\). Hence \( C \) is \((2,1)\)-extendable by Lemma 1.1. □
Proof of Theorem 1.3. There is an \((i, j)\)-plane \(\pi\) satisfying \(\pi \cap F_d = \emptyset\) if \(\mathcal{C}\) is (2,1)-extendable. From Table 2 and Table 3 in [3], any \((\Phi_0, \Phi_1)\) flat with \((\Phi_0, \Phi_1) \neq (40, 36)\) contains an \((i, j)\)-plane with \((i, j) = (4, 9)\) or \((13, 0)\). Hence \(\mathcal{C}\) is (2,1)-extendable by Lemma 1.1.

\(\square\)

Proof of Theorem 1.4. ("only if" part:) Assume that \(\mathcal{C}\) is (2,1)-extendable. Then there is an \((i, j)\)-plane \(\pi\) satisfying \(\pi \cap F_d = \emptyset\). From Table 2 and Table 3 in [3], an \((i, j)\)-plane in the \((40, 36)_4\) flat \(\Sigma\) satisfies \((i, j) \in \{(4, 0), (1, 6), (4, 3), (4, 6), (7, 3)\}\). The condition \((C_4 - 0)\) holds if \((i, j) \in \{(4, 0), (1, 6), (4, 3), (7, 3)\}\) and the condition \((C_4 - 3)\) holds if \((i, j) = (4, 6)\).

("if" part:) Assume that the condition \((C_4 - 0)\) holds. Let \(\ell\) be a \((1, 0)\)-plane satisfying \(\ell \setminus F \subset F_e\). Then, it can be proved that there is a \((7, 3)\)-plane through \(\ell\) in the \((40, 36)_4\) flat \(\Sigma\). Hence, \(\mathcal{C}\) is (2,1)-extendable by Lemma 1.1.

Assume that The condition \((C_4 - 3)\) holds. Then the plane \(\delta = \langle Q_1, Q_2, Q_3 \rangle\) forms a \((4, 6)\)-plane satisfying \(\delta \cap F_d = \emptyset\).

\(\square\)

Proof of Theorem 1.6. There is an \((i, j)\)-solid \(\pi\) satisfying \(\pi \cap F_d = \emptyset\) if \(\mathcal{C}\) is (2,1)-extendable. From Table 3 and Table 4 in [3], any \((\Phi_0, \Phi_1)\) flat with \((\Phi_0, \Phi_1) \notin \{(121, 108), (112, 126), (130, 117)\}\) contains an \((i, j)\)-solid with \((i, j) = (13, 27)\) or \((40, 0)\). Hence \(\mathcal{C}\) is (2,1)-extendable by Lemma 1.1.

\(\square\)

Proof of Theorem 1.7. ("only if" part:) Assume that \(\mathcal{C}\) is (2,1)-extendable. Then there is an \((i, j)\)-solid \(\pi\) satisfying \(\pi \setminus F \subset F_e\). From Table 3 and Table 4 in [3], an \((i, j)\)-solid in the \((121, 108)_5\) flat \(\Sigma\) satisfies \((i, j) \in \{(13, 0), (4, 18), (13, 9), (10, 15), (16, 12), (13, 18), (22, 9)\}\). The condition \((C_5 - 1)\) holds if \((i, j) = (4, 18)\) or \((22, 9)\). The conditions \((C_5 - 3)\), \((C_5 - 4)\), \((C_5 - 5)\), \((C_5 - 6)\) hold if \((i, j) = (13, 18), (13, 9), (16, 12), (10, 15)\), respectively.

("if" part:) Assume that the condition \((C_5 - 1)\) holds. Then there exist a \((4, 0)\)-plane \(\delta_1\) and a \((4, 9)\)-plane \(\delta_2\) such that \(\ell = \delta_1 \cap \delta_2\) is a \((4, 0)\)-line and that \(\delta_1 \setminus \ell \subset F_e\). Since \(\Delta = (\delta_1, \delta_2)\) is necessarily a \((22, 9)\)-solid in the \((121, 108)_5\) flat \(\Sigma\), we have \(\Delta \cap F_d = \emptyset\). Hence \(\mathcal{C}\) is (2,1)-extendable. Similarly, the conditions \((C_5 - 3)\), \((C_5 - 4)\), \((C_5 - 5)\), \((C_5 - 6)\) imply the existence of an \((i, j)\)-solid \(\Delta\) with \((i, j) = (13, 18), (13, 9), (16, 12), (10, 15)\), respectively, satisfying \(\Delta \cap F_d = \emptyset\).

\(\square\)

Theorems 1.8 and 1.9 can be proved similarly to Theorem 1.7.

3. Proof of Theorem 1.5.

Assume that \(\mathcal{C}\) is not (2,1)-extendable. Then no three points of \(F_e\) are collinear by Theorem 1.4. Thus, \(F_e\) forms a \textit{cap} and hence \(|F_e| \leq 20\) since the largest size of a cap in \(\text{PG}(4, 3)\) is 20. Every 20-cap in \(\text{PG}(4, 3)\) is either a \(\Gamma\)-\textit{cap} or \(\Delta\)-\textit{cap} ([2]).
Theorem 3.1([2]). Let $E$ be a 10-cap in a solid $H$ of $PG(4,3)$. Let $P_1, \ldots, P_{10}$ be the points of $E$ and let $V$ be a point of $PG(4,3) \setminus H$. Then the set consisting any two of the three points different from $V$ on each of the 10 lines $\langle V, P_i \rangle$ forms a 20-cap. Such a cap is called a $\Gamma$-cap or a cap of type $\Gamma$.

Theorem 3.2([2]). Let $C_1, C_2, C_3, C_4$ be the points of a 4-arc in a plane $\pi$ in $PG(4,3)$. Let $Z_1, Z_2$, and $Z_3$ be the points $\langle C_1, C_2 \rangle \cap \langle C_3, C_4 \rangle$, $\langle C_2, C_3 \rangle \cap \langle C_4, C_1 \rangle$, and $\langle C_2, C_4 \rangle \cap \langle C_1, C_3 \rangle$ respectively. Let $Z_{i1}, Z_{i2}$ be the points on the line $\langle Z_j, Z_k \rangle$ ($j,k \neq i$) other than $Z_j$ and $Z_k$. Let $L = \{V_1, V_2, V_3, V_4\}$ be a line skew to $\pi$. Then the set of points $C_1, C_2, C_3, C_4$ together with the points on the following lines $\langle A, B \rangle$ other than $A$ and $B$:

\[
\langle V_i, Z_j \rangle \ (i = 1, 2; \ j = 2, 3), \ \langle V_3, Z_3h \rangle \ (h = 1, 2), \ \langle V_4, Z_2h \rangle \ (h = 1, 2)
\]

forms a 20-cap. Such a cap is called a $\Delta$-cap or a cap of type of $\Delta$.

For $i = 1, 2$, a point $P \in F_1$ is called a focal point of a hyperplane $H$ if the following three conditions hold:

(f-1) $\langle P, Q \rangle$ is a $(0,2)$-line for $Q \in F_1 \cap H$,

(f-2) $\langle P, Q \rangle$ is a $(2,1)$-line for $Q \in F_{3-i} \cap H$,

(f-3) $\langle P, Q \rangle$ is a $(1,6-3i)$-line for $Q \in F_0 \cap H$.

Such a hyperplane $H$ is called a focal hyperplane of $P$. We also employ the following theorem.

Theorem 3.3([8]). In a $(40,36)_4$-flat, every point of $F_1$ (resp. $F_2$) has the unique focal $(10,15)$-solid (resp. $(16,12)$-solid), and vice versa.

We show that $F_e$ cannot form a 20-cap. Then $|F_e| = |F_2| - |F_{3e}| = 45 - A_d/2 < 20$, giving Theorem 1.5. First, suppose that $F_e$ is a 20-cap of type $\Gamma$ in Theorem 3.1.

If $V \in F_0$, then the line $\langle V, P_i \rangle$ is necessarily a $(1,0)$-line for $i = 1, \ldots, 10$. This contradicts that there are exactly six $(1,0)$-lines through a fixed point of $F_0$ in the $(40,36)_4$-flat $\Sigma$.

If $V \in F_1$, then $\langle V, P_i \rangle$ is a $(0,2)$-line for $i = 1, \ldots, 10$. Let $H'$ be the focal solid of $V$. Then $H'$ is a $(10,15)$-solid by Theorem 3.3. Let $E'$ be the projection of $E$ from $V$ onto $H'$ if $H \neq H'$, otherwise let $E' = E$. Then $E' \subset F_1$. Since $E$ is an elliptic quadric, so is $E'$. Hence, there are exactly 10 planes (resp. 30 planes) in $H'$ meeting $E'$ in one point (resp. four points) in $H'$. On the other hand, the spectrum of a $(10,15)$-solid is $c^{(3)}_{16}, c^{(3)}_{43}, c^{(3)}_{46} = (10,15,15)$. Hence there are at most $c^{(3)}_{16} + c^{(3)}_{46} = 25$ planes in $H$ meeting $E$ in four points, a contradiction.

If $V \in F_2$, then $\langle V, P_i \rangle$ is a $(1,0)$-line for $i = 1, \ldots, 10$. Let $H'$ be a $(16,12)$-solid which is the focal solid of $V$, and let $E'$ be the projection of $E$ from $V$
onto $H'$ as the previous case. Then $E' \subset F_0$. Since $F_0 \cap H'$ is a hyperbolic quadric in $H'$, $F_0 \cap H'$ cannot contain the 10-cap $E'$, a contradiction. Thus, $F_e$ cannot form a $\Gamma$-cap.

Next, suppose that $F_e$ is a 20-cap of type $\Delta$ in Theorem 3.2. Since $C_1, \ldots, C_4$ are points of $F_e$ in the case, $\pi$ is a $(4,0)$-plane or a $(1,6)$-plane or a $(4,3)$-plane.

Assume $\pi$ is a $(4,0)$-plane. Then, one of the lines $\langle Z_2, Z_3 \rangle$, $\langle Z_1, Z_2 \rangle$, $\langle Z_1, Z_3 \rangle$ must be a $(4,0)$-line. If $\langle Z_2, Z_3 \rangle$ is a $(4,0)$-line, then $\langle V_1, Z_3 \rangle$, $\langle V_2, Z_3 \rangle$, $\langle V_3, Z_{31} \rangle$, and $\langle V_4, Z_{21} \rangle$ are $(1,0)$-lines, and $L$ is a $(2,0)$-line, a contradiction. One can get a contradiction similarly for other cases.

Assume $\pi$ is a $(1,6)$-plane. Since $C_1, \ldots, C_4$ form a 4-arc contained in two $(1,0)$-lines of $\pi$, one of $Z_1, Z_2, Z_3$ must be the point $\pi \cap F_0$. Suppose $Z_3 \in F_0$. Then $\langle Z_3, C_i \rangle$ is a $(1,0)$-line for $i = 1, 2$ and $\langle Z_3, Z_j \rangle$ is a $(1,3)$-line for $j = 1, 2$. Since $\langle V_1, Z_3 \rangle$ is a $(1,0)$-line, we have $V_1 \in F_2$, so $\langle V_1, Z_4 \rangle$ is a $(0,1)$-line, a contradiction. One can get a contradiction similarly if $Z_1 \in F_0$ or $Z_2 \in F_0$.

Assume $\pi$ is a $(4,3)$-plane. If $Z_3$ is a point of $F_0$, then $\langle Z_3, C_1 \rangle$ and $\langle Z_3, C_2 \rangle$ are $(1,0)$-lines, which contradicts that there is only one $(1,0)$-line through a fixed point of $F_0$ in a $(4,3)$-plane. One can get a contradiction similarly if $Z_3 \in F_1$ or $Z_3 \in F_2$. Thus, $F_e$ cannot form a $\Delta$-cap as well.

□

References


