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On the $(2,1)$-extendability of ternary linear codes

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Abstract. We show that every $[n,k,d]_3$ code with diversity (Φ_0,Φ_1), $3 \le k \le 5$, $gcd(d,3)=1$, is $(2,1)$-extendable except for the case $(\Phi_0,\Phi_1)=(40,36)$ for $k=5$, and that an $[n,5,d]_3$ code with diversity $(40,36)$, $gcd(d,3)=1$, is $(2,1)$-extendable if $A_d \le 50$. Geometric conditions for the $(2,1)$-extendability of not necessarily extendable $[n,k,d]_3$ codes for $k=5,6$ are also given.

1 Introduction

Let $\mathbb{F}_q^n$ denote the vector space of n-tuples over $\mathbb{F}_q$, the field of q elements. A linear code $\mathcal{C}$ of length n, dimension k and minimum (Hamming) distance d over $\mathbb{F}_q$ is referred to as an $[n,k,d]_q$ code. The *weight* of a vector $\boldsymbol{x} \in \mathbb{F}_q^n$, denoted by $wt(\boldsymbol{x})$, is the number of nonzero coordinate positions in $\boldsymbol{x}$. The weight distribution of $\mathcal{C}$ is the list of numbers A_i which is the number of codewords of $\mathcal{C}$ with weight i. The weight distribution with $(A_0, A_d, ...) = (1, \alpha, ...)$ is also expressed as $0^1 d^\alpha \cdots$. We only consider *non-degenerate* codes having no coordinate which is identically zero.

For an $[n,k,d]_q$ code $\mathcal{C}$ with a generator matrix G, $\mathcal{C}$ is called (l,s)*-extendable* (to $\mathcal{C}'$) if there exist l vectors $h_1,\dots,h_l \in \mathbb{F}_q^k$ so that the extended matrix $[G, h_1^{\mathrm{T}}, \cdots, h_l^{\mathrm{T}}]$ generates an $[n+l,k,d+s]_q$ code $\mathcal{C}'$ ([1]). Then $\mathcal{C}'$ is called an (l,s)*-extension* of $\mathcal{C}$. '$(1,1)$-extendable' is simply called *extendable*. In this paper we are concerned with (2,1)-extendability of ternary linear codes with dimension $k \le 6$.

Let $\mathcal{C}$ be an $[n,k,d]_3$ code with $k \ge 3$, $\gcd(3,d)=1$. The *diversity* (Φ_0,Φ_1) of $\mathcal{C}$ is given as the pair of integers:

$$\Phi_0 = \frac{1}{2}\sum_{3|i, i\neq 0} A_i, \quad \Phi_1 = \frac{1}{2}\sum_{i \not\equiv 0,d \pmod 3} A_i,$$

where the notation $x|y$ means that x is a divisor of y.

Let $\mathcal{D}_k$ be the set of all possible diversities of $\mathcal{C}$. $\mathcal{D}_k$ has been determined in [3] for $k \le 6$ and in [5] for $k \ge 7$. For $k \ge 3$, let $\mathcal{D}_k^*$ and $\mathcal{D}_k^+$ be as follows:

$$\mathcal{D}_k^* = \{(\theta_{k-2}, 0), (\theta_{k-3}, 2 \cdot 3^{k-2}), (\theta_{k-2}, 2 \cdot 3^{k-2}), (\theta_{k-2} + 3^{k-2}, 3^{k-2})\}, \quad \mathcal{D}_k^+ = \mathcal{D}_k \setminus \mathcal{D}_k^*,$$

where $\theta_j = (3^{j+1} - 1)/2$. It is known that $\mathcal{D}_k^*$ is included in $\mathcal{D}_k$ and that $\mathcal{C}$ is extendable if $(\Phi_0, \Phi_1) \in \mathcal{D}_k^*$ ([3]). The necessary and sufficient conditions for the extendability of $\mathcal{C}$ with $(\Phi_0, \Phi_1) \in \mathcal{D}_k^+$ are given in [3-7] for $k \le 6$.

We denote by $\mathrm{PG}(r, q)$ the projective geometry of dimension r over $\mathbb{F}_q$. A *j-flat* is a projective subspace of dimension j in $\mathrm{PG}(r, q)$. 0-flats, 1-flats, 2-flats, 3-flats, $(r-2)$-flats and $(r-1)$-flats are called *points, lines, planes, solids, secundums* and *hyperplanes*, respectively.

Let $\mathcal{C}$ be an $[n, k, d]_q$ code with a generator matrix $G = [g_1, \cdots, g_k]^{\mathrm{T}}$. For $P = \mathbf{P}(p_1, \cdots, p_k) \in \Sigma$, *the weight of P with respect to $\mathcal{C}$* is defined in [4] as

$$w_{\mathcal{C}}(P) = wt(\sum_{i=1}^{k} p_i g_i).$$

¿From now on, let $\mathcal{C}$ be an $[n, k, d]_3$ code with $\gcd(d, 3) = 1$, $k \ge 3$. Let

$$\begin{aligned}
F_0 &= \{P \in \Sigma \mid w_{\mathcal{C}}(P) \equiv 0 \pmod 3\},\\
F_2 &= \{P \in \Sigma \mid w_{\mathcal{C}}(P) \equiv d \pmod 3\},\\
F_d &= \{P \in \Sigma \mid w_{\mathcal{C}}(P) = d\}, \quad F_e = F_2 \setminus F_d\\
F_1 &= \Sigma \setminus (F_0 \cup F_2), \quad F = F_0 \cup F_1.
\end{aligned}$$

Lemma 1.1([4]). *$\mathcal{C}$ is $(2,1)$-extendable iff there exist two hyperplanes H_1, H_2 of Σ such that $F_d \cap H_1 \cap H_2 = \emptyset$. Equivalently, $F \cup F_e$ contains a secundum of Σ.*

We give the necessary and sufficient conditions for the (2,1)-extendability of $[n, k, d]_3$ codes with diversity (Φ_0, Φ_1), $3 \le k \le 6$, $d \equiv 1$ or $2 \pmod 3$ from this geometrical point of view.

A t-flat Π of Σ with $|\Pi \cap F_0| = i$, $|\Pi \cap F_1| = j$ is called an *$(i, j)_t$ flat.* An $(i, j)_1$ flat is called an *(i, j)-line*. An *(i, j)-plane* and an *(i, j)-solid* are defined similarly. We denote by $\mathcal{F}_j$ the set of j-flats of Σ. Let Λ_t be the set of all possible (i, j) for which an $(i, j)_t$ flat exists in Σ. Then we have
$\Lambda_1 = \{(1,0), (0,2), (2,1), (1,3), (4,0)\}$,
$\Lambda_2 = \{(4,0), (1,6), (4,3), (4,6), (7,3), (4,9), (13,0)\}$,
$\Lambda_3 = \{(13,0), (4,18), (13,9), (10,15), (16,12), (13,18), (22,9), (13,27), (40,0)\}$,
$\Lambda_4 = \{(40,0), (13,54), (40,27), (31,45), (40,36), (40,45), (49,36), (40,54), (67,27),$

$(40,81),(121,0)\}$,
$\Lambda_5 = \{(121,0),(40,162),(121,81),(94,135),(121,108),(112,126),(130,117),$
$(121,135),(148,108),(121,162),(202,81),(121,243),(364,0)\}$,

see [3]. Let $\Pi_t \in \mathcal{F}_t$. Denote by $c_{i,j}^{(t)}$ the number of $(i,j)_{t-1}$ flats in Π_t and let ${\varphi_s}^{(t)} = |\Pi_t \cap F_s|$, $s = 0, 1$. $({\varphi_0}^{(t)}, {\varphi_1}^{(t)})$ is called the *diversity of* Π_t and the list of $c_{i,j}^{(t)}$'s is called its *spectrum.* Thus Λ_t is the set of all possible diversities of Π_t.

According to the diversity of $\mathcal{C}$ we give the necessary and sufficient conditions for the (2,1)-extendability of $\mathcal{C}$ using some of the following six conditions:

For $k \geq 4$, let (C_k-0), (C_k-1) and (C_k-2) be the following conditions:
($\mathbf{C}_k$-0) there exists a $(\theta_{k-4}, 0)_{k-3}$ flat δ_1 in Σ satisfying $\delta_1 \setminus F_0 \subset F_e$;

($\mathbf{C}_k$-1) (C_k-0) holds and there exists a $(\theta_{k-4}, 3^{k-3})_{k-3}$ flat δ_2 in Σ such that $\delta_1 \cap \delta_2$ is a $(\theta_{k-4}, 0)_{k-4}$ flat;
($\mathbf{C}_k$-2) there exist two $(\theta_{k-4}, 0)_{k-3}$ flats δ_1, δ_2 in Σ such that $\delta_1 \cap \delta_2$ is a $(\theta_{k-4}, 0)_{k-4}$ flat with $(\delta_1 \cup \delta_2) \setminus (\delta_1 \cap \delta_2) \subset F_e$;

For $k \geq 4$, let (C_k-3) and (C_k-4) be the following conditions:
($\mathbf{C}_4$-3) there are three non-collinear points $Q_1, Q_2, Q_3 \in F_e$ such that the three lines $\langle Q_1, Q_2 \rangle$, $\langle Q_2, Q_3 \rangle$, $\langle Q_3, Q_1 \rangle$ are $(0,2)$-lines;
($\mathbf{C}_k$-3) there exist three $(\theta_{k-5}, 0)_{k-4}$ flats δ_1, δ_2, δ_3 through a fixed $(\theta_{k-5}, 0)_{k-5}$ flat L such that $\langle \delta_1, \delta_2 \rangle$, $\langle \delta_2, \delta_3 \rangle$, $\langle \delta_3, \delta_1 \rangle$ form distinct $(\theta_{k-5}, 2 \cdot 3^{k-4})_{k-3}$ flats and that $(\delta_1 \cup \delta_2 \cup \delta_3) \setminus L \subset F_e$ holds;

($\mathbf{C}_4$-4) there are three non-collinear points $P_1, P_2, P_3 \in F_1$ such that the three lines $\langle P_1, P_2 \rangle$, $\langle P_2, P_3 \rangle$, $\langle P_3, P_1 \rangle$ are $(0,2)$-lines each of which contains two points of F_e;
($\mathbf{C}_k$-4) there exist a $(\theta_{k-5}, 0)_{k-5}$ flat L, three $(\theta_{k-5}, 3^{k-4})_{k-4}$ flats δ_1', δ_2', δ_3' through L, and six $(\theta_{k-5}, 0)_{k-4}$ flats $\delta_1, \cdots, \delta_6$ through L such that $\langle \delta_i', \delta_j' \rangle$ forms a $(\theta_{k-5}, 2 \cdot 3^{k-4})_{k-3}$ flat containing two of $\delta_1, \cdots, \delta_6$ for $1 \leq i < j \leq 3$ and that $(\cup_{i=1}^{6} \delta_i) \setminus L \subset F_\mathrm{e}$ holds.

For $k = 5$, let (C_k-5) and (C_k-6) be the following conditions:
($\mathbf{C}_5$-5) there exist a (4,0)-line l and four skew (1,0)-lines l_1, l_2, l_3, l_4 such that each of $l_1, ..., l_4$ meets l and that $\langle l_1, l_2, l_3, l_4 \rangle \in \mathcal{F}_3$ and $(\cup_{i=1}^{4} l_i) \setminus l \subset F_e$ hold;

($\mathbf{C}_5$-6) there exists a (4,3)-plane δ in Σ and a point $R \in F_e$ such that $l_i = \langle R, P_i \rangle$ is a $(1,0)$-line for $i = 1,2,3,4$ and $(\delta \cup l_1 \cup l_2 \cup l_3 \cup l_4) \cap F_d = \emptyset$, where $F_0 \cap \delta = \{P_1, \ldots, P_4\}$.

Theorem 1.2. *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with diversity (Φ_0, Φ_1), $k = 3$ or 4, $gcd(3,d) = 1$. Then $\mathcal{C}$ is $(2,1)$-extendable.*

Theorem 1.3. *Let $\mathcal{C}$ be an $[n,5,d]_3$ code with diversity $(\Phi_0, \Phi_1) \neq (40,36)$, $gcd(3,d) = 1$. Then $\mathcal{C}$ is $(2,1)$-extendable.*

Theorem 1.4. *Let $\mathcal{C}$ be an $[n, 5, d]_3$ code with diversity $(40, 36)$, gcd$(3, d) = 1$. Then $\mathcal{C}$ is $(2, 1)$-extendable iff one of the conditions* (C_4-0), (C_4-3) *holds.*

Theorem 1.5. *Let $\mathcal{C}$ be an $[n, 5, d]_3$ code with diversity $(40, 36)$, gcd$(3, d) = 1$. Then $\mathcal{C}$ is $(2, 1)$-extendable if $A_d \leq 50$.*

Theorem 1.6. *Let $\mathcal{C}$ be an $[n, 6, d]_3$ code with diversity $(\Phi_0, \Phi_1) \notin \{(121, 108), (112, 126), (130, 117)\}$, gcd$(3, d) = 1$. Then $\mathcal{C}$ is $(2, 1)$-extendable.*

Theorem 1.7. *Let $\mathcal{C}$ be an $[n, 6, d]_3$ code with diversity $(121, 108)$, gcd$(3, d) = 1$. Then $\mathcal{C}$ is $(2, 1)$-extendable iff one of the conditions* (C_5-1), (C_5-3), (C_5-4), (C_5-5), (C_5-6) *holds.*

Theorem 1.8. *Let $\mathcal{C}$ be an $[n, 6, d]_3$ code with diversity $(112, 126)$, gcd$(3, d) = 1$. Then $\mathcal{C}$ is $(2, 1)$-extendable iff one of the conditions* (C_5-2), (C_5-3), (C_5-4), (C_5-5), (C_5-6) *holds.*

Theorem 1.9. *Let $\mathcal{C}$ be an $[n, 6, d]_3$ code with diversity $(130, 117)$, gcd$(3, d) = 1$. Then $\mathcal{C}$ is $(2, 1)$-extendable iff one of the conditions* (C_5-0), (C_5-3), (C_5-4), (C_5-5), (C_5-6) *holds.*

Example. Let $\mathcal{C}$ be a $[15, 5, 8]_3$ code with a generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 1 \end{bmatrix},$$

whose weight distribution is $0^1 8^{60} 9^{40} 10^{62} 11^{20} 12^{40} 13^{10} 14^{10}$ (diversity (40,36)). Then we can take $Q_1 = (0, 0, 1, 1, 1)$, $Q_2 = (1, 2, 1, 2, 2)$, $Q_3 = (1, 1, 0, 0, 1)$ so that the condition (C_4-3) of Theorem 1.4 holds. Since $V(x_0 + 2x_1 + 2x_2 + x_3) \cap V(x_0 + x_2 + 2x_4) = \langle Q_1, Q_2, Q_3 \rangle$, by adding the column $(1, 2, 2, 1, 0)^{\mathrm{T}}$ and $(1, 0, 1, 0, 2)^{\mathrm{T}}$ to G, we get a (2,1)-extension of $\mathcal{C}$ whose weight distribution is $0^1 9^{38} 10^{56} 11^{46} 12^{34} 13^{30} 14^{26} 15^8 16^4$.

2. Proof of Theorems 1.2–1.4, 1.6–1.9.

Proof of Theorem 1.2. When $k = 3$, there is a point $P \notin F_d$ iff $\mathcal{C}$ is (2,1)-extendable. Obviously, any plane have such a point P in F. When $k = 4$, there is an (i, j)-line l with $l \cap F_d = \emptyset$ iff $\mathcal{C}$ is (2,1)-extendable. ¿From Table 1 and Table 2 in [3], it can be checked that any solid has an (i, j)-line with $(i, j) = (1, 3)$ or $(4, 0)$. Hence $\mathcal{C}$ is (2,1)-extendable by Lemma 1.1. □

Proof of Theorem 1.3. There is an (i, j)-plane π satisfying $\pi \cap F_d = \emptyset$ iff $\mathcal{C}$ is (2,1)-extendable. From Table 2 and Table 3 in [3], any $(\Phi_0, \Phi_1)_4$ flat with $(\Phi_0, \Phi_1) \neq (40, 36)$ contains an (i, j)-plane with $(i, j) = (4, 9)$ or $(13, 0)$. Hence $\mathcal{C}$ is (2,1)-extendable by Lemma 1.1. $\square$

Proof of Theorem 1.4. ("only if" part:) Assume that $\mathcal{C}$ is (2,1)-extendable. Then there is an (i, j)-plane π satisfying $\pi \cap F_d = \emptyset$. From Table 2 and Table 3 in [3], an (i, j)-plane in the $(40, 36)_4$ flat Σ satisfies $(i, j) \in \{(4, 0), (1, 6), (4, 3), (4, 6), (7, 3)\}$. The condition ($C_4$-0) holds if $(i, j) \in \{(4, 0), (1, 6), (4, 3), (7, 3)\}$ and the condition (C_4-3) holds if $(i, j) = (4, 6)$.
("if" part:) Assume that the condition (C_4-0) holds. Let l be a $(1, 0)$-line satisfying $l \setminus F \subset F_e$. Then, it can be proved that there is a $(7, 3)$-plane through l in the $(40, 36)_4$ flat Σ. Hence, $\mathcal{C}$ is (2,1)-extendable by Lemma 1.1.
Assume that The condition (C_4-3) holds. Then the plane $\delta = \langle Q_1, Q_2, Q_3 \rangle$ forms a $(4, 6)$-plane satisfying $\delta \cap F_d = \emptyset$. $\square$

Proof of Theorem 1.6. There is an (i, j)-solid π satisfying $\pi \cap F_d = \emptyset$ iff $\mathcal{C}$ is (2,1)-extendable. From Table 3 and Table 4 in [3], any $(\Phi_0, \Phi_1)_5$ flat with $(\Phi_0, \Phi_1) \notin \{(121, 108), (112, 126), (130, 117)\}$ contains an (i, j)-solid with $(i, j) = (13, 27)$ or (40,0). Hence $\mathcal{C}$ is (2,1)-extendable by Lemma 1.1. $\square$

Proof of Theorem 1.7. ("only if" part:) Assume that $\mathcal{C}$ is (2,1)-extendable. Then there is an (i, j)-solid π satisfying $\pi \setminus F \subset F_e$. From Table 3 and Table 4 in [3], an (i, j)-solid in the $(121, 108)_5$ flat Σ satisfies $(i, j) \in \{(13, 0)$, (4,18), (13,9), (10,15), (16,12), (13,18), (22,9)$\}$. The condition (C_5-1) holds if $(i, j) = (4, 18)$ or (22,9). The conditions (C_5-3), (C_5-4), (C_5-5), (C_5-6) hold if $(i, j) = (13, 18)$, (13,9), (16,12), (10,15), respectively.
("if" part:) Assume that the condition (C_5-1) holds. Then there exist a (4,0)-plane δ_1 and a (4,9)-plane δ_2 such that $l = \delta_1 \cap \delta_2$ is a (4,0)-line and that $\delta_1 \setminus l \subset F_e$. Since $\Delta = \langle \delta_1, \delta_2 \rangle$ is necessarily a (22,9)-solid in the $(121, 108)_5$ flat Σ, we have $\Delta \cap F_d = \emptyset$. Hence $\mathcal{C}$ is (2,1)-extendable. Similarly, the conditions (C_5-3), (C_5-4), (C_5-5), (C_5-6) imply the existence of an (i, j)-solid Δ with $(i, j) = (13, 18)$, (13,9), (16,12), (10,15), respectively, satisfying $\Delta \cap F_d = \emptyset$. $\square$

Theorems 1.8 and 1.9 can be proved similarly to Theorem 1.7.

3. Proof of Theorem 1.5.

Assume that $\mathcal{C}$ is not $(2, 1)$-extendable. Then no three points of F_e are collinear by Theorem 1.4. Thus, F_e forms a *cap* and hence $|F_e| \leq 20$ since the largest size of a cap in PG(4, 3) is 20. Every 20-cap in PG(4, 3) is either a Γ-*cap* or Δ-*cap* ([2]).

Theorem 3.1([2]). *Let E be a* 10*-cap in a solid H of* $\mathrm{PG}(4,3)$*. Let $P_1, \cdots, P_{10}$ be the points of E and let V be a point of* $\mathrm{PG}(4,3) \setminus H$*. Then the set consisting any two of the three points different from V on each of the 10 lines $\langle V, P_i\rangle$ forms a 20-cap. Such a cap is called a Γ-cap or a cap of type Γ.*

Theorem 3.2([2]). *Let C_1, C_2, C_3, C_4 be the points of a* 4*-arc in a plane π in* $\mathrm{PG}(4,3)$*. Let Z_1, Z_2 and Z_3 be the points $\langle C_1, C_2\rangle \cap \langle C_3, C_4\rangle, \langle C_2, C_3\rangle \cap \langle C_4, C_1\rangle$ and $\langle C_2, C_4\rangle \cap \langle C_1, C_3\rangle$ respectively. Let Z_{i1}, Z_{i2} be the points on the line $\langle Z_j, Z_k\rangle (j, k \neq i)$ other than Z_j and Z_k. Let $L = \{V_1, V_2, V_3, V_4\}$ be a line skew to π. Then the set of points C_1, C_2, C_3, C_4 together with the points on the following lines $\langle A, B\rangle$ other than A and B:*

$$\langle V_i, Z_j\rangle \ (i = 1, 2;\ j = 2, 3),\ \langle V_3, Z_{3h}\rangle \ (h = 1, 2),\ \langle V_4, Z_{2h}\rangle \ (h = 1, 2)$$

forms a 20*-cap. Such a cap is called a Δ-cap or a cap of type of Δ.*

For $i = 1, 2$, a point $P \in F_i$ is called a *focal point* of a hyperplane H if the following three conditions hold:
(f-1) $\langle P, Q\rangle$ is a $(0, 2)$-line for $Q \in F_i \cap H$,
(f-2) $\langle P, Q\rangle$ is a $(2, 1)$-line for $Q \in F_{3-i} \cap H$,
(f-3) $\langle P, Q\rangle$ is a $(1, 6 - 3i)$-line for $Q \in F_0 \cap H$.
Such a hyperplane H is called a *focal hyperplane* of P. We also employ the following theorem.

Theorem 3.3([8]). *In a* $(40, 36)_4$*-flat, every point of F_1* (*resp.* F_2) *has the unique focal* $(10, 15)$*-solid* (*resp.* $(16, 12)$*-solid*)*, and vice versa.*

We show that F_e cannot form a 20-cap. Then $|F_e| = |F_2| - |F_d| = 45 - A_d/2 < 20$, giving Theorem 1.5. First, suppose that F_e is a 20-cap of type Γ in Theorem 3.1.

If $V \in F_0$, then the line $\langle V, P_i\rangle$ is necessarily a $(1, 0)$-line for $i = 1, \cdots, 10$. This contradicts that there are exactly six $(1, 0)$-lines through a fixed point of F_0 in the $(40, 36)_4$-flat Σ.

If $V \in F_1$, then $\langle V, P_i\rangle$ is a $(0, 2)$-line for $i = 1, \cdots, 10$. Let H' be the focal solid of V. Then H' is a $(10, 15)$-solid by Theorem 3.3. Let E' be the projection of E from V onto H' if $H \neq H'$, otherwise let $E' = E$. Then $E' \subset F_1$. Since E is an elliptic quadric, so is E'. Hence, there are exactly 10 planes (resp. 30 planes) in H' meeting E' in one point (resp. four points) in H'. On the other hand, the spectrum of a $(10, 15)$-solid is $(c^{(3)}_{1,6}, c^{(3)}_{4,3}, c^{(3)}_{4,6}) = (10,15,15)$. Hence there are at most $c^{(3)}_{1,6} + c^{(3)}_{4,6} = 25$ planes in H meeting E in four points, a contradiction.

If $V \in F_2$, then $\langle V, P_i\rangle$ is a $(1, 0)$-line for $i = 1, \cdots, 10$. Let H' be a $(16, 12)$-solid which is the focal solid of V, and let E' be the projection of E from V

onto H' as the previous case. Then $E' \subset F_0$. Since $F_0 \cap H'$ is a hyperbolic quadric in H', $F_0 \cap H'$ cannot contain the 10-cap E', a contradiction. Thus, F_e cannot form a Γ-cap.

Next, suppose that F_e is a 20-cap of type Δ in Theorem 3.2. Since $C_1, \ldots, C_4$ are points of F_e in the case, π is a $(4,0)$-plane or a $(1,6)$-plane or a $(4,3)$-plane.

Assume π is a $(4,0)$-plane. Then, one of the lines $\langle Z_2, Z_3\rangle$, $\langle Z_1, Z_2\rangle$, $\langle Z_1, Z_3\rangle$ must be a $(4,0)$-line. If $\langle Z_2, Z_3\rangle$ is a $(4,0)$-line, then $\langle V_1, Z_3\rangle$, $\langle V_2, Z_3\rangle$, $\langle V_3, Z_{31}\rangle$, and $\langle V_4, Z_{21}\rangle$ are $(1,0)$-lines, and L is a $(2,0)$-line, a contradiction. One can get a contradiction similarly for other cases.

Assume π is a $(1,6)$-plane. Since $C_1, \ldots, C_4$ form a 4-arc contained in two (1,0)-lines of π, one of Z_1, Z_2, Z_3 must be the point $\pi \cap F_0$. Suppose $Z_3 \in F_0$. Then $\langle Z_3, C_i\rangle$ is a (1,0)-line for $i = 1, 2$ and $\langle Z_3, Z_j\rangle$ is a (1,3)-line for $j = 1, 2$. Since $\langle V_1, Z_3\rangle$ is a $(1,0)$-line, we have $V_1 \in F_2$, so $\langle V_1, Z_2\rangle$ is a $(0,1)$-line, a contradiction. One can get a contradiction similarly if $Z_1 \in F_0$ or $Z_2 \in F_0$.

Assume π is a $(4,3)$-plane. If Z_3 is a point of F_0, then $\langle Z_3, C_1\rangle$ and $\langle Z_3, C_2\rangle$ are $(1,0)$-lines, which contradicts that there is only one (1,0)-line through a fixed point of F_0 in a $(4,3)$-plane. One can get a contradiction similarly if $Z_3 \in F_1$ or $Z_3 \in F_2$. Thus, F_e cannot form a Δ-cap as well. □

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