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Sets of mutually orthogonal resolutions of BIBDs[1]

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Abstract. The nonisomorphic sets of m mutually orthogonal resolutions (m-MORs) of doubly resolvable $2-(v,k,\lambda)$ designs with small parameters are constructed and lower bounds on the number of m-MORs of multiple designs are obtained.

1 Introduction

For the basic concepts and notations concerning combinatorial designs and their resolutions refer, for instance, to [2], [3], [7].

Let $V = \{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathcal{B} = \{B_j\}_{j=1}^{b}$ − a finite collection of k-element subsets of V, called *blocks*. If any 2-subset of V is contained in exactly λ blocks of $\mathcal{B}$, then $D = (V,\mathcal{B})$ is a 2-(v,k,λ) *design*, or *balanced incomplete block design (BIBD)*. We shall call two blocks B_1 and B_2 equal ($B_1 = B_2$) if they are incident with the same set of points.

Two designs are *isomorphic* if there exists a one-to-one correspondence between the point and block sets of the first design and respectively, the point and block sets of the second design, and if this one-to-one correspondence does not change the incidence. An *automorphism* is an isomorphism of the design to itself, i.e. a permutation of the points that transforms the blocks into blocks.

A 2-(v,k,$m\lambda$) design is called an *m-fold multiple* of 2-(v,k,λ) designs if there is a partition of its blocks into m subcollections $\mathcal{B}_1$, $\mathcal{B}_2$, ... $\mathcal{B}_m$, which form 2-(v,k,λ) designs $D_1, D_2, ..., D_m$. If $D_1 = D_2 = ... = D_m$ we call the design *true m-fold multiple* of D_1.

A *resolution* of the design is a partition of the collection of blocks into *parallel classes*, such that each point is in exactly one block of each parallel class. We shall call two parallel classes of the resolution $\mathcal{R}$, R_1 and R_2 equal ($R_1 = R_2$) if each block of R_1 is equal to a block of R_2. The design is *resolvable* if it has at least one resolution. Two resolutions are isomorphic if there exists

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an automorphism of the design transforming each parallel class of the first resolution into a parallel class of the second one.

There is a one-to-one correspondence [5] between the resolutions of $2-(qk,k,\lambda)$ designs and the $(r,qk,r-\lambda)_q$ equidistant codes, where $r=\lambda(qk-1)/(k-1)$ and $q>1$.

Two resolutions $\mathcal{R}$ and $\mathcal{T}$ of one and the same design are *orthogonal* if the number of blocks in $R_i \cap T_j$ is either 0 or 1 for all $1 \le i,j \le r$. Orthogonal resolutions may or may not be isomorphic to each other. A *doubly resolvable design (DRD)* is a design which has at least two orthogonal resolutions. We denote by *ROR* a resolution which is orthogonal to at least one other resolution, by m*-MOR* a set of m mutually orthogonal resolutions, and by m*-MORs* sets of m mutually orthogonal resolutions. Two m-MORs are isomorphic if there is an automorphism of the design transforming them into each other. The m-MOR is maximal if no more resolutions can be added to it.

The newest results and an extended bibliography and summary of previous works on the existence of DRDs can be found in [1] and a method for construction and classification of RORs and DRDs in [6].

The aim of the present work is the classification up to isomorphism of m-MORs of 2-(v,k,λ) DRDs with small parameters and the establishment of some lower bounds on their number for multiple designs.

2 m-MORs construction and classification

We start with a DRD and construct its resolutions block by block. For each resolution $\mathcal{R}$ we check if it is isomorphic to a lexicographically smaller one, and if not, we try to construct another resolution $\mathcal{R}_1$, which is lexicographically greater than $\mathcal{R}$ and orthogonal to it. We next repeat the same for $\mathcal{R}_1$, $\mathcal{R}_2$, etc, constructing at each step a resolution $\mathcal{R}_m$ orthogonal to all the resolutions $\mathcal{R}, \mathcal{R}_1, ..., \mathcal{R}_{m-1}$, and checking if this m-MOR is isomorphic to a lexicographically smaller one. We output a new m-MOR if it is maximal.

The results are summarized in Table 1, where the last column shows the number of the design in the tables [4] and a/b means that the number of nonisomorphic MORs is b, a of them maximal.

3 m-MORs of multiple designs

We first recall definitions and notations concerning sets of orthogonal Latin squares (see for instance [3]).

Table 1: Classification of inequivalent m-MORs

q	v	k	λ	b	r	DRDs	RORs	2-MORs	3-MORs	4-MORs	No
2	6	3	8	40	20	1	1	1/1	-	-	236
2	6	3	12	60	30	1	1	0/1	1/1	-	596
2	6	3	16	80	40	1	1	0/$\geq$485	0/$\geq$485	$\geq$485/$\geq$485	1078
2	8	4	6	28	14	1	1	1/1	-	-	101
2	8	4	9	42	21	1	1	0/1	1/1	-	278
2	8	4	12	56	28	4	4	7/17	0/60	60/60	524
2	10	5	16	72	36	5	5	5/5	-	-	891
2	10	5	24	108	54	6	6	2/7	5/5	-	-
2	12	6	10	44	22	1	1	1/1	-	-	319
2	12	6	15	66	33	1	1	0/1	1/1	-	743
2	12	6	20	88	44	546	546	691/$\geq$718	0/$\geq$27	$\geq$27/$\geq$27	-
2	16	8	14	60	30	5	5	5/5	-	-	618
2	16	8	21	90	45	5	5	0/5	5/5	-	-
2	20	10	18	76	38	3	3	3/3	-	-	1007
3	9	3	3	36	12	3	5	2/7	5/5	-	66
3	9	3	4	48	16	38	83	388/495	333/334	1/1	145
4	12	3	2	44	11	20	70	319/321	1/2	1/1	55
4	16	4	2	40	10	1	1	0/1	1/1	-	44

A *Latin square of side (order)* n is an $n \times n$ array in which each cell contains a single symbol from an n-set S, such that each symbol occurs exactly once in each row and exactly once in each column. A Latin square exists for any integer side n. An $m \times n$ *Latin rectangle* is an $m \times n$ array in which each cell contains a single symbol from an n-set S, such that each symbol occurs exactly once in each row and at most once in each column. An $m \times n$ Latin rectangle can always be completed to a Latin square of side n.

Let L be a Latin square of side n on symbol set E_3 with rows indexed by the elements of the n-set E_1 and columns indexed by the elements of the n-set E_2. Let $\tau = \{(x_1, x_2, x_3) : L(x_1, x_2) = x_3\}$. Let $\{a, b, c\} = \{1, 2, 3\}$. The (a, b, c)-conjugate of L, $L_{(a,b,c)}$ has rows indexed by E_a, columns by E_b, and symbols by E_c, and is defined by $L_{(a,b,c)}(x_a, x_b) = x_c$ for each $(x_1, x_2, x_3) \in \tau$.

Two Latin squares L_1 and L_2 are *equivalent (isotopic)* if there are three bijections from the rows, columns and symbols of L_1 to the rows, columns and symbols, respectively of L_2 that map L_1 to L_2. L_1 and L_2 are *main class equivalent* if L_1 is equivalent to any conjugate of L_2.

Two Latin squares of side n $L_1 = (a_{ij})$ on symbol set S_1 and $L_2 = (b_{ij})$ on symbol set S_2 are *orthogonal* if every element in $S_1 \times S_2$ occurs exactly once among the n^2 pairs $(a_{ij}, b_{ij}), i, j = 1, 2, ..., n$. A set of Latin squares of side n, $L_1, L_2, ..., L_m$ is *mutually orthogonal(a set of MOLS)*, if L_i and L_j are orthogonal for $i, j = 1, 2, ..., n, i \neq j$. A set of MOLS of side n can have at most $n - 1$ elements.

Different types of equivalence of MOLS can be defined. In this paper we use the following definitions of conjugates and equivalence of MOLS:

Let $\mathcal{M}$ be a set of m MOLS $L_1, L_2, ..., L_m$ of side n on symbol sets respectively $E_3, E_4, ..., E_{m+2}$ and with rows and columns indexed by the elements of the n-sets E_1 and E_2 respectively. Let $\tau = \{(x_1, x_2, ..., x_{m+2}) : L_i(x_1, x_2) = x_{i+2}, i = 1, 2, ..., m\}$. Let $\{a_1, a_2, ..., a_{m+2}\} = \{1, 2, ..., m+2\}$. The $(a_1, a_2, ..., a_{m+2})$ conjugate of $\mathcal{M}$, $\mathcal{M}_{(a_1,a_2,...,a_{m+2})}$ contains the Latin squares $L_i : L_i(a_1, a_2) = a_{i+2}$, $i = 1, 2, ..., m$ for each $(x_1, x_2, ..., x_{m+2}) \in \tau$.

Two sets of MOLS $\mathcal{M}_1$ and $\mathcal{M}_2$ are *equivalent (isotopic)* if there are three bijections from the rows, columns and symbols of $\mathcal{M}_1$ to the rows, columns and symbols, respectively of $\mathcal{M}_2$ that map $\mathcal{M}_1$ to $\mathcal{M}_2$. $\mathcal{M}_1$ and $\mathcal{M}_2$ are *main class equivalent* if $\mathcal{M}_1$ is equivalent to any conjugate of $\mathcal{M}_2$.

Proposition 3.1 *Let D be a 2-(v,k,λ) design and $v = 2k$.*

1) D is doubly resolvable iff it is resolvable and each set of k points is either incident with no block, or with at least two blocks of the design.

2) If D is doubly resolvable and at least one set of k points is in m blocks, and the rest in 0 or more than m blocks, then D has at least one maximal m-MOR, no i-MORs for $i > m$ and no maximal i-MORs for $i < m$.

The proof is based on:

1) If one block of a parallel class is known, the point set of the second one is known too. Suppose D has m-MOR $\mathcal{R}_1, \mathcal{R}_2, ...\mathcal{R}_m$. Consider a block with exactly $p - 1$ equal blocks. Denote by $1, 2, ..., p$ the parallel classes of $\mathcal{R}_1$, in which these blocks are, the blocks themselves by $1_1, 2_1, ..., p_1$ and the second blocks in the classes by $1_2, 2_2, ..., p_2$. Since block i_1 should be with block j_2 $(i, j = 1, 2, ..., p)$ at most once in a parallel class of the m-MOR, the class numbers of the second blocks form an $m \times p$ Latin rectangle. An example for $p = 4$ and $m = 3$ is presented in Fig.1.

2) A $2 \times m$ Latin rectangle can be completed to a Latin square of order m.

Proposition 3.2 *Let $l_{q-1,m}$ be the number of main class inequivalent sets of $q - 1$ MOLS of side m. Let $q = v/k$ and $m \geq q$. Let the 2-(v,k,mλ) design D be a true m-fold multiple of a resolvable 2-(v,k,λ) design d. If $l_{q-1,m} > 0$, then*

Figure 1: 4 equal parallel classes of 3 mutually orthogonal resolutions, $v = 2k$

	1	2	3	4		*Latin rectangle*			
$\mathcal{R}_1$	$1_1 1_2$	$2_1 2_2$	$3_1 3_2$	$4_1 4_2$		1	2	3	4
$\mathcal{R}_2$	$1_1 2_2$	$2_1 1_2$	$3_1 4_2$	$4_1 3_2$	$\Longrightarrow$	2	1	4	3
$\mathcal{R}_3$	$1_1 3_2$	$2_1 4_2$	$3_1 1_2$	$4_1 2_2$		3	4	1	2

D is doubly resolvable and has at least $\binom{\frac{r}{m} - 1 + l_{q-1,m}}{\frac{r}{m}}$ *m-MORs.*

The proof is based on:

Consider a resolution $\mathcal{R}_1$ of D, such that each parallel class of $\mathcal{R}_1$ is equal to a parallel class of the resolution $\mathcal{T}$ of d. We can partition the collection of parallel classes of $\mathcal{R}_1$ into subcollections $P_1, P_2, ..., P_{r/m}$ of size m, such that the classes in a subcollection are equal. m-MOR containing $\mathcal{R}_1$ can be constructed as follows: the first block of each class equals the first block of the corresponding class of $\mathcal{R}_1$ and the other blocks of P_i form a set $\mathcal{M}_i$ of $q-1$ MOLS of side m. An example for $m = 4$ and $q = 3$ is presented in Fig. 2a.

Figure 2: 4 equal parallel classes of 4 mutually orthogonal resolutions, $v = 3k$

a)relation to a set $\mathcal{M}$ of two MOLS of side 4

	1	2	3	4		$\mathcal{M} = \mathcal{M}_{(1,2,3,4)}$							
$\mathcal{R}_1$	$1_1 1_2 1_3$	$2_1 2_2 2_3$	$3_1 3_2 3_3$	$4_1 4_2 4_3$		1	2	3	4	1	2	3	4
$\mathcal{R}_2$	$1_1 2_2 3_3$	$2_1 1_2 4_3$	$3_1 4_2 1_3$	$4_1 3_2 2_3$	$\Longrightarrow$	2	1	4	3	3	4	1	2
$\mathcal{R}_3$	$1_1 3_2 4_3$	$2_1 4_2 3_3$	$3_1 1_2 2_3$	$4_1 2_2 1_3$		3	4	1	2	4	3	2	1
$\mathcal{R}_4$	$1_1 4_2 2_3$	$2_1 3_2 1_3$	$3_1 2_2 4_3$	$4_1 1_2 3_3$		4	3	2	1	2	1	4	3

b)automorphism α transforming first blocks into second blocks

	1	2	3	4
$\mathcal{R}_1$	$1_2 1_1 1_3$	$2_2 2_1 2_3$	$3_2 3_1 3_3$	$4_2 4_1 4_3$
$\mathcal{R}_2$	$1_2 2_1 3_3$	$2_2 1_1 4_3$	$3_2 4_1 1_3$	$4_2 3_1 2_3$
$\mathcal{R}_3$	$1_2 3_1 4_3$	$2_2 4_1 3_3$	$3_2 1_1 2_3$	$4_2 2_1 1_3$
$\mathcal{R}_4$	$1_2 4_1 2_3$	$2_2 3_1 1_3$	$3_2 2_1 4_3$	$4_2 1_1 3_3$

c)relation to $\mathcal{M}_{(1,3,2,4)}$ - the $(1,3,2,4)$ conjugate of S

	1	2	3	4		$\mathcal{M}_{(1,3,2,4)}$							
$\mathcal{R}_1$	$1_1 1_2 1_3$	$2_1 2_2 2_3$	$3_1 3_2 3_3$	$4_1 4_2 4_3$		1	2	3	4	1	2	3	4
$\mathcal{R}_2$	$1_1 2_2 4_3$	$2_1 1_2 3_3$	$3_1 4_2 2_3$	$4_1 3_2 1_3$	$\Longrightarrow$	2	1	4	3	4	3	2	1
$\mathcal{R}_3$	$1_1 3_2 2_3$	$2_1 4_2 1_3$	$3_1 1_2 4_3$	$4_1 2_2 3_3$		3	4	1	2	2	1	4	3
$\mathcal{R}_4$	$1_1 4_2 3_3$	$2_1 3_2 4_3$	$3_1 2_2 1_3$	$4_1 1_2 2_3$		4	3	2	1	3	4	1	2

Permutation of design classes, numbers of equal classes, or resolutions of the m-MOR invokes respectively permutation of columns, symbols and rows of all Latin squares in $\mathcal{M}_i$. A nontrivial point automorphism α can invoke a transformation of $\mathcal{M}_i$ into one of its conjugates (an example is presented in Fig. 2b,c.) or into a conjugate of $\mathcal{M}_j, i, j = 1, 2, ..., r/m, i \neq j$. Thus there are at least $l_{q-1,m}$ inequivalent ways to fix $\mathcal{M}_i$.

The number of different ways to choose u integers $i_1, i_2, ...i_u$, such that $i_1 + i_2 + ... + i_u = w$ is $Q(u, w) = \binom{u+w-1}{w}$.

Corollary 3.3 *Let l_m be the number of main class inequivalent Latin squares of side m. Let $v/k = 2$ and $m \geq 2$. Let the 2-(v,k,mλ) design D be a true m-fold multiple of a 2-(v,k,λ) design d, and let d be resolvable, but not doubly resolvable. Then D is doubly resolvable and has at least* $\binom{\frac{r}{m} - 1 + l_m}{\frac{r}{m}}$ *m-MORs, no maximal i-MORs for $i < m$ and no i-MORs for $i > m$.*

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