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Doubles of Hadamard 2-(15,7,3) designs

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Abstract. Nonisomorphic 2-(15,7,6) designs which are doubles of Hadamard 2-(15,7,3) designs are constructed. The automorphism groups of the Hadamard designs are considered to reduce the number of isomorphic ones among the constructed doubles. Canonical form of the incidence matrices is used to reject isomorphic designs and to establish the order of their automorphism groups. There are 5 non isomorphic 2-(15,7,3) designs H_1, H_2, H_3, H_4 and H_5. All doubles of H_1 and H_i are classified for $i = 1, 2, 3, 4$ and 5.

1 Introduction

Basic definitions. For the basic concepts and notations concerning combinatorial designs refer, for instance, to [1], [2], [14].

Let $\mathcal{P} = \{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathcal{B} = \{B_j\}_{j=1}^{b}$ a finite collection of k-element subsets of $\mathcal{P}$, called *blocks*. $D = (\mathcal{P}, \mathcal{B})$ is a *design* with parameters t-(v,k,λ) if any t-subset of $\mathcal{P}$ is contained in exactly λ blocks of $\mathcal{B}$. Any point $P_i \in \mathcal{P}$ occurs in the same number r of blocks of $\mathcal{B}$. If $v = b$ the design is *symmetric* and $r = k$ too. A symmetric 2-($4m-1$,$2m-1$,$m-1$) design is called a Hadamard 2-design.

Two designs D_1 and D_2 are *isomorphic* ($D_1 \sim D_2$) if there exists a one-to-one correspondence between the point and block sets of the first design and the point and block sets of the second design, and if this one-to-one correspondence does not change the incidence. Isomorphic designs are indistinguishable by algebraical means. In some cases, however, it is very important to distinguish isomorphic, but different designs. We then speak about *labelled* designs (see for instance [3]) and mean that the points are ordered in some way.

An *automorphism* is an isomorphism of the design to itself. The set of all automorphisms of a design forms a group called its *full group of automorphisms.* Each subgroup of this group is a group of automorphisms of the design.

Each 2-(v,k,λ) design determines the existence of 2-$(v,k,2\lambda)$ designs. These 2-$(v,k,2\lambda)$ designs are called *quasidoubles* of 2-(v,k,λ) designs. A quasidouble 2-$(v,k,2\lambda)$ design is *reducible* into two 2-(v,k,λ) designs if there is a partition of its blocks into two subcollections each of which forms a 2-(v,k,λ) design. A reducible quasidouble is called a *double.*

We denote the set $\{1, 2, ..., v\}$ by N_v, the symmetric group of all permutations of N_v by S_v, the full automorphism group of a design D by $Aut(D)$, and a double design which is reducible to the designs D_1 and D_2 by $[D_1 \parallel D_2]$.

Incidence matrix of a labeled 2-(v,k,λ) design is a $(0,1)$ matrix with v rows and b columns, where the element of the i-th row ($i \in N_v$) and j-th column ($j \in N_b$) is 1 if the i-th point of $\mathcal{P}$ occurs in the j-th block of $\mathcal{B}$ and 0 otherwise. The design is completely determined by its incidence matrix. The incidence matrices of two isomorphic designs are equivalent.

Let us denote the incidence matrix of a design D by $\mathcal{D}$. Define standard lexicographic order relations on the rows and columns of $\mathcal{D}$. We denote by $\mathcal{D}^{sort}$ a column-sorted matrix obtained from $\mathcal{D}$ by sorting the columns in decreasing order. Define a standard lexicographic order on the matrices considering each matrix as an ordered v-tuple of the v rows. Let $\mathcal{D}^{max} = max\{\varphi\mathcal{D}^{sort} : \varphi \in S_v\}$ (corresponds to the notation *romim* [13] about the incidence matrix of a graph). $\mathcal{D}^{max}$ is a canonical form of the incidence matrix $\mathcal{D}$.

Other notations. Let $D = (\mathcal{P}, \mathcal{B})$ be a $2-(v,k,\lambda)$ design, $G \subseteq S_v$ and the permutation $\sigma = (\sigma_1, \sigma_2, ..., \sigma_v) \in G$.

$\sigma P_i = P_{\sigma_i}$,

$\sigma D = (\sigma\mathcal{P}, \sigma\mathcal{B}) : B = \{P_{i_j}\}_{j=1}^{k} \in \mathcal{B} \Leftrightarrow \sigma B = \{P_{\sigma_{i_j}}\}_{j=1}^{k} \in \sigma\mathcal{B}$.

$\sigma D_1 = D_2 \;\Leftrightarrow D_1 \sim D_2$.

$P^G = \{\sigma P : \sigma \in G\}$ (*orbit* of P with respect to G).

$G_P = \{\sigma : \sigma \in G, \sigma P = P\}$ (*stabiliser* of the point P with respect to G).

$G_{\{i_1,i_2,...,i_m\}} = \{\sigma \in G : \forall j \in N_m, \sigma P_{i_j} = P_{i_j}\} = \bigcap G_{P_{i_j}}$ for $m \in N_v$ is the *stabiliser of the point set* $\{P_{i_1}, P_{i_2}, ..., P_{i_m}\}$ with respect to G.

Hadamard 2-(15,7,3) designs. There are five nonisomorphic 2-(15,7,3) designs. We denote them by $H_1, H_2, ..., H_5$ such that $\forall i \in N_4 :\;\; \mathcal{H}_i^{max} > \mathcal{H}_{i+1}^{max}$.

The full automorphism groups of H_1, H_2, H_3, H_4 and H_5 are of orders 20160, 576, 96, 168 and 168 respectively. We use automorphisms and point orbits of these groups to decrease the number of constructed isomorphic designs. The number of isomorphic but distinguished 2-(15,7,3) designs is
$15! \sum_{n=1}^{5} \frac{1}{Aut(D_n)} = 31524292800$.

The present work. Subject of the present work are 2-(15,7,6) designs, which are reducible into two Hadamard 2-(15,7,3) designs H_1 and $H_i, i = 1,2,...,5$. Their block collection is obtained as a union of the block collections of H_1 and φH_i, $\forall\varphi \in S_v$. The action of $Aut(H_1)$ and $Aut(H_i)$ is considered and doubles are not constructed for part of the permutations of S_v, because it is shown that they lead to isomorphic doubles.

Transformation of matrices in some canonical form is used by many authors for the rejection of equivalent solutions (see for instance [4], [6] and [7]). In the present work classification of the obtained designs is made by the help of $\mathcal{D}^{max}$.

There exist at least 57810 nonisomorphic 2-(15,7,6) designs [12]. This lower bound is improved in [10] and [11], where all 2-(15,7,6) designs with automorphisms of prime odd orders were constructed, their number was determined to be 92 323 and 12 786 of them were found to be reducible. Here a classification of all 2-(15,7,6) designs reducible into H_1 and H_i, $i = 1, 2, 3, 4, 5$ is presented. The results coincide with those in [10] and [11] and improve the lower bound to 1566454. A further classification of all reducible 2-(15,7,6) designs is also of interest for setting higher lower bounds on the number of Hadamard designs of greater parameters [8], [9].

2 Doubles of 2-(15,7,3) designs

Preliminaries. Consider a 2-(15,7,6) design $D = [D' \parallel D'']$. Without loss of generality we can assume that the first 15 columns of the incidence matrix $\mathcal{D}$ form a sub-matrix equal to $\mathcal{D}'$ and the next 15 columns form $\mathcal{D}''$. In this case we will write $D = D' \parallel D''$ instead of $[D' \| D'']$.

The number of doubles $H_1 \parallel \varphi H_i, i = 1, ..., 5$, is greater than $4,7.10^{12}$. Our purpose is to construct exactly one representative of each isomorphism class. That is why it is very important to show which permutations applied to H_i lead to isomorphic designs and skip them.

The construction algorithm is based on the next simple proposition.

Proposition 1 *Let D' and D'' be two $2-(v, k, \lambda)$ designs and let α' and α'' be automorphisms of D' and D'' respectively. Then for all permutations $\varphi \in S_v$ the double designs $[D' \parallel \varphi D'']$, $[D' \parallel \varphi\alpha'' D'']$ and , $[D' \parallel \alpha'\varphi D'']$ are isomorphic.*

Proof. $\forall \alpha' \in Aut(D') \;\Rightarrow\; [D' \parallel \alpha'\varphi D''] \sim \alpha'^{-1}[D' \parallel \alpha'\varphi D''] = [D' \parallel \varphi D'']$ and $\forall \alpha'' \in Aut(D'') \;\Rightarrow\; [D' \parallel \varphi\alpha'' D''] = [D' \parallel \varphi D'']$.

Corolary 1 *If the double design $[D' \parallel \varphi D'']$ is already constructed, then all permutations in the set $Aut(D')\varphi \bigcup \varphi Aut(D'') \setminus \{\varphi\}$ can be omitted.*

We implement that with a back-track search algorithm.

Let the last considered permutation be $\varphi = (\varphi_1, \varphi_2, ..., \varphi_v)$. The next lexicographically greater than it permutation $\psi = (\psi_1, \psi_2, ..., \psi_v)$ is formed in the following way:

We look for the greatest $m \in N_{v-1} \bigcup \{0\}$, such that

- if $i \in N_m$ then $\varphi_i = \psi_i$ and $\varphi_{m+1} < \psi_{m+1},\ \psi_{m+1} \in N_v \backslash \{\varphi_1, \varphi_2, ..., \varphi_m\}$.
- The number ψ_{m+1} is taken from the set N''_m that contains a unique representative of each of the orbits of the permutation group $Aut(D'')_{\{\varphi_1,\varphi_2,...,\varphi_m\}}$.
- If $j \in N_m$ and $\psi_j > \psi_{m+1}$ then points P'_j and P'_{m+1} should not be in one orbit with respect to the stabilizer $Aut(D')_{\{1,2,...,j-1\}}$.

The isomorphism test is applied when a new double design D is constructed by the help of the canonical D^{max} form of its incidence matrix. The algorithm finding D^{max} gives as additional effect the full automorphism group of D.

3 Classification results

The number of nonisomorphic reducible 2-(15,7,6) designs from the five cases $H_1 \parallel H_i,\ i \in N_5$ is 1566454. Their classification with respect to the order of the automorphism groups is presented in Table 1.

A double design can have automorphisms of order 2 and automorphisms which preserve the two constituent designs (see for instance [5]). That is why among the constructed designs are all reducible 2-(15,7,6) designs with automorphisms of order 5. Their number is 6 and is the same as in [10].

Table 1: Order of the automorphism group of $H_1 \parallel H_i,\ i = 1, 2, 3, 4, 5$.

| *Aut. gr.* |	1	2	3	4	6	7	8	9
Designs	1 559 007	5 012	990	173	119	15	860	1
| *Aut. gr.* |	10	12	14	16	18	21	24	32
Designs	4	32	4	61	1	5	48	6
| *Aut. gr.* |	36	42	48	56	64	96	120	168
Designs	1	2	14	3	6	3	1	2
| *Aut. gr.* |	192	288	336	384	576	2048	2688	20160
Designs	4	1	1	4	1	1	1	1

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