Eleventh International Workshop on Algebraic and Combinatorial Coding Theory
June 16-22, 2008, Pamporovo, Bulgaria *pp. 199-202*

Optimality of the trivial (28,8,2,3) superimposed code

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Abstract. In this paper we prove that the trivial $(28, 8, 2, 3)$ superimposed code is optimal.

1 Introduction

Definition 1 *A binary $N \times T$ matrix $C = (c_{ij})$ is called an (N, T, w, r) superimposed code (SIC) if for any pair of subsets $W, R \subset \{1, 2, \dots, T\}$ such that $|W| = w$, $|R| = r$ and $W \cap R = \varnothing$ there exists a row $i \in \{1, 2, \dots, N\}$ such that $c_{ij} = 1$ for all $j \in W$ and $c_{ij} = 0$ for all $j \in R$. We say also that C is a (w, r) superimposed code of length N and size T.*

The trivial code is a simple example for an (N, T, w, r) superimposed code. The length N of the trivial code is $\binom{T}{w}$ and its rows are all possible binary vectors of weight w.

Let $N(T, w, r)$ is the minimum length of an (N, T, w, r) superimposed code for given values of T, w and r. The code is called optimal when $N = N(T, w, r)$. The exact values of $N(T, 2, 3)$ are known for $T \leq 7$.

T	5	6	7
$N(T, 2, 3)$	10	15	21

The trivial $(10, 5, 2, 3)$, $(15, 6, 2, 3)$ and $(21, 7, 2, 3)$ superimposed codes are optimal. Kim and Lebedev [2] have proved that $24 \leq N(8, 2, 3) \leq 28$ and $26 \leq N(9, 2, 3) \leq 30$. Therefore the trivial $(36, 9, 2, 3)$ superimposed code is not optimal. In this paper we prove the nonexistence of $(27, 8, 2, 3)$ superimposed code. Consequently the trivial $(28, 8, 2, 3)$ superimposed code is optimal.

2 Preliminaries

Definition 2 *Two (N, T, w, r) superimposed codes are equivalent if one of them can be transformed into the other by a permutation of the rows and a permutation of the columns.*

[1]Partially supported by the Technical University of Gabrovo under Grant C-801/2008.

Let C be a binary $N \times T$ matrix. Denote by $d(x, y)$ the Hamming distance between two columns x and y and by S_x and S_y – the characteristic sets of the columns x and y respectively. The following lemma is obvious.

Lemma 3 $d(x, y) = |S_x| + |S_y| - 2|S_x \cap S_y|$.

Let $d(x, y, z) = d(x, y) + d(x, z) + d(y, z)$. From Lemma 3 we obtain $d(x, y, z) = d(x, y) + d(x, z) + d(y, z) = 2(|S_x| + |S_y| + |S_z| - |S_x \cap S_y| - |S_x \cap S_z| - |S_y \cap S_z|)$. Consequently $d(x, y, z)$ is even number. Denote by $d_2 = \min\{d(x, y) \mid x, y \in C,\ x \neq y\}$ and by $d_3 = \min\{d(x, y, z) \mid x, y, z \in C,\ x \neq y, x \neq z, y \neq z\}$. It is clear that $3d_2 \leq d_3$. Let $d(C) = \sum_{x,y \in C,\ x \neq y} d(x, y)$.

Lemma 4 *(Plotkin bound) [3]* $\binom{T}{2} d_2 \leq d(C) \leq N \left\lfloor \frac{T}{2} \right\rfloor \left\lfloor \frac{T+1}{2} \right\rfloor$.

Corollary 5 $\binom{T}{3} d_3 \leq (T-2)d(C) \leq (T-2)N \left\lfloor \frac{T}{2} \right\rfloor \left\lfloor \frac{T+1}{2} \right\rfloor$.

Definition 6 *Let $x_1, x_2, ..., x_k$ be different columns of the superimposed code C. The residual code $Res(C, x_1 = v_1, x_2 = v_2, ..., x_k = v_k)$ of C is the code obtained by taking all the rows in which C has value v_i in the column x_i for $i = 1, 2, ..., k$ and deleting the columns $x_1, x_2, ..., x_k$ in the selected rows.*

Lemma 7 *Suppose C is an (N, T, w, r) superimposed code and x and y are two different columns of C. Then*
(a) $Res(C, x = 1)$ is a $(|S_x|, T-1, w-1, r)$ SIC;
(b) $Res(C, x = 0)$ is an $(N - |S_x|, T-1, w, r-1)$ SIC.

Lemma 8 *[2] $N(6, 1, 2) = 6$ and $N(7, 2, 2) = 14$.*

Lemma 9 *[1] Any $(6, 6, 1, 2)$ superimposed code is equivalent to the trivial $(6, 6, 1, 2)$ superimposed code*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Using computer programs for generation of $(1, 2)$ and $(2, 3)$ superimposed codes and for code equivalence we proved the following two lemmas:

Lemma 10 *Any $(7, 6, 1, 2)$ superimposed code is equivalent to one of the codes*

$$C_{1,2,\ldots,7} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ * & * & * & * & * & * \end{pmatrix} \quad C_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The last row of $C_{1,2,\ldots,7}$ *is* 0000000*,* 0000001*,* 0000011*,* 0000111*,* 0001111*,* 0011111*,* 0111111 *or* 1111111 *respectively.*

Lemma 11 *Any* $(21, 7, 2, 3)$ *SIC is equivalent to the trivial* $(21, 7, 2, 3)$ *SIC.*

3 The nonexistence of (27,8,2,3) SIC

Lemma 12 *Let* C *be a* $(27, 8, 2, 3)$ *superimposed code. Then* $d_2 \geq 12$*.*

Proof. Let x and y be two different columns in C. Since $N(6, 1, 2) = 6$ (Lemma 8), $|S_x \cap \overline{S}_y| \geq 6$ and $|\overline{S}_x \cap S_y| \geq 6$. Therefore $d(x, y) \geq 12$ and $d_2 \geq 12$. □

Lemma 13 *Let* C *be a* $(27, 8, 2, 3)$ *superimposed code and* x *and* y *are two different columns of* C*. Then* $Res(C, x = 0, y = 1)$ *contains at most 5 rows of weight* 0 *or* 1*.*

Proof. Suppose the matrix C contains at least 6 rows of weight 0 or 1. Let C' be the matrix obtained of C by deleting of the column y. C' is a $(27, 7, 2, 3)$ superimposed code and contains 6 rows of weight 0 or 1. Consequently the remaining 21 rows of C' form a $(21, 7, 2, 3)$ superimposed code. According to Lemma 11 this code is equivalent to the trivial $(21, 7, 2, 3)$ superimposed code, hence all its rows are of weight 2. Therefore $d(C') \leq 21 \times 10 + 6 \times 6 = 246$. According to Lemma 12 the distance between any two columns of C' is at least 12. It follows from Lemma 4 that $d(C') \geq \binom{7}{2}.12 = 252$, which is a contradiction. Therefore $Res(C, x = 0, y = 1)$ contains at most 5 rows of weight 0 or 1. □

Lemma 14 *Let* C *be a* $(27, 8, 2, 3)$ *superimposed code. Then* $d_2 = 14$*.*

Proof. Let x and y be two columns of C for which $d(x, y) = d_2$. It follows from Lemma 13 that the length of each of the codes $Res(C, x = 0, y = 1)$ and $Res(C, x = 1, y = 0)$ is at least 7, hence $d_2 \geq 14$. According to Lemma 4 $d_2 \leq 15$. Consequently one of this residual codes is of length 7 and is equivalent to the code C_8 of Lemma 10. Therefore $d(C) \leq 429$. It follows from Corollary 5 that $d_3 \leq 45\frac{27}{28}$. But d_3 is an even number, hence $d_3 \leq 44$. Consequently $d_2 = 14$.

□

Theorem 15 *There is no* $(27,8,2,3)$ *superimposed code.*

Proof. Let C be a (27,8,2,3) superimposed code. It follows from Lemma 14 that there exist two columns x and y such that $d(x,y)=14$. Hence the residual codes $Res(C,x=0,y=1)$ and $Res(C,x=1,y=0)$ are equivalent to the code C_8 of Lemma 10. We can write C as follows:

<table>
<tr><td>x y</td><td></td></tr>
<tr><td>0 1
⋮ ⋮
0 1</td><td>(7,6,1,2) SIC</td></tr>
<tr><td>1 0
⋮ ⋮
1 0</td><td>(7,6,1,2) SIC</td></tr>
<tr><td>0 0
⋮ ⋮
0 0</td><td>M rows</td></tr>
<tr><td>1 1
⋮ ⋮
1 1</td><td>13 − M rows</td></tr>
</table>

Using a computer program we obtained that there are exactly 30 inequivalent possibilities for the first 14 rows of C. $Res(C,x=0)$ is an $(M+7,7,2,2)$ SIC. According to Lemma 8 $M\geq 7$. C is a $(27,8,2,3)$ SIC, hence $M\leq 12$.

Using a computer program we constructed the missing part column by column, checking at each step the condition of Lemma 14, the superimposed code property and the sorted last 13 rows property.

It turned out that the extension to a $(27,8,2,3)$ superimposed code is impossible. Therefore there is no $(27,8,2,3)$ superimposed code. □

Theorem 16 *The trivial* $(28,8,2,3)$ *superimposed code is optimal.*

References

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