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Jacket conference matrices and Paley transformation

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Abstract. Paley construction can be generalized into two ways. First we apply it to construct Jacket matrices, staring from new type of matrices and second generalization is to increase the dimensions to obtain (bigger) Jacket matrices. In this paper we describe a modified Paley construction to produce Jacket matrices that can be denoted as a new type of matrices called "Jacket Conference matrices".

1 Introduction

Jacket matrices [1], [2] are defined to be $n \times n$ matrices $J = ||j_{i,k}||$ over a field F with the property $JJ' = n.I_n$, where $J' = ||j'_{i,k}||$ is the transpose matrix of the element inverse of J, i.e., $j'_{i,k} = (j_{k,i})^{-1}$. These matrices are used in Digital Signal Processing and Coding theory. In [3] Paley constructed two classes of Hadamard matrices (known as Paley type 1 and Paley type 2). In this paper we describe a modified Paley construction to produce Jacket matrices that can be denoted as a new type of matrices called "Jacket Conference matrices".

In Section 2, we present the Paley method [3] in more general form, in order to apply it for constructing Hadamard matrices from symmetric Conference matrices.

2 Paley transformation

Recall that the Kronecker product, $A \otimes B$, of two matrices $[A]_n = ||a_{i,j}||$ and $[B]_m = ||b_{i,j}||$ is defined as

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix} \otimes B := \begin{pmatrix} a_{1,1}B & a_{1,2}B & \dots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \dots & a_{2,n}B \\ \vdots & \vdots & & \vdots \\ a_{m,1}B & a_{m,2}B & \dots & a_{m,n}B \end{pmatrix}$$

In the above formula $a_{ij}B$ stands for a sub-matrix (not for an element).

Let C be an $m \times m$ symmetric Conference matrix. Paley statement says, that for

$$[A]_2 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}_2, \quad [B]_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_2,$$

the matrix

$$[S]_{2m} := [A]_2 \otimes [C]_m + [B]_2 \otimes [I]_m$$

is Hadamard [3].

Proposition 1. *Let C be an $m \times m$ symmetric Conference matrix, and A and B be $n \times n$ Hadamard matrices. The matrix*

$$[S]_{nm} := [A]_n \otimes [C]_m + [B]_n \otimes [I]_m \tag{1}$$

is an Hadamard matrix if and only if

$$[A]_n[B]_n^t + [B]_n[A]_n^t = 0, \tag{2}$$

where $[X]^t$ is the transpose matrix of $[X]$.

Proof. To check if $[S]_{mn}$ is an Hadamard matrix we must show, that $[S]_{mn}[S]_{mn}^t = mn[I]_{mn}$. Since

$$[S]_{mn}^t = [A]_n^t \otimes [C]_m^t + [B]_n^t \otimes [I]_m,$$

we have

$$[S]_{mn}[S]_{mn}^t = ([A]_n \otimes [C]_m + [B]_n \otimes [I]_m)([A]_n^t \otimes [C]_m^t + [B]_n^t \otimes [I]_m) =$$

$$= ([A]_n \otimes [C]_m)([A]_n^t \otimes [C]_m^t) + ([A]_n \otimes [C]_m)([B]_n^t \otimes [I]_m)+$$

$$+([B]_n \otimes [I]_m)([A]_n^t \otimes [C]_m^t) + ([B]_n[B]_n^t) \otimes [I]_m =$$

$$= ([A]_n \otimes [C]_m)([A]_n^t \otimes [C]_m^t) + ([B]_n[B]_n^t) \otimes [I]_m + ([A]_n[B]_n^t) \otimes [C]_m + ([B]_n[A]_n^t) \otimes [C]_m^t$$

$[C]_m^t = [C]_m$, and $[A]_n$ and $[C]_n$ are orthogonal matrices, thus $[A]_n \otimes [C]_m$ is also orthogonal matrix, and $[B]_n$ is Hadamard matrix. Hence

$$[S]_{mn}[S]_{mn}^t = (mn - n)[I]_{mn} + n[I]_{mn} + ([A]_n[B]_n^t + [B]_n[A]_n^t) \otimes [C]_m.$$

To get $[S]_{mn}$ to be Hadamard matrix (i.e. $[S]_{mn}[S]_{mn}^t = mn[I]_{mn}$) would be equivalent to

$$[A]_n[B]_n^t + [B]_n[A]_n^t = 0.$$

Note that in the case $n = 2$ Proposition 1 gives the Paley construction. Also, the above proposition motivate us to give the following definition.

Definition 1. *The pair $([A]_n, [B]_n)$ of Hadamard matrices of order n is called matched, iff the equation (2) holds.*

3 Jacket conference matrices

In this section instead of using transpose matrix of $[X]_n$, we can use transpose matrix of element inverse, and we shall denote it with $[X]'_n$ (where $x'_{ij} = (x_{ji})^{-1}$). We preserve matrices $[A]_n$ and $[B]_n$ to be Hadamard, but $[C]_m$ must be selected properly, and $[C]'_m$ must be defined in little bit different way.

Definition 2. *We shall call the $n \times n$ square matrix A is a Jacket Conference matrix, if the following conditions hods:*

1. $a_{i,i} = 0$, *for* $i = 1, 2, \ldots, n$.

2. $a_{i,j} \neq 0$, *for all* $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$.

3. $\sum_{s \in \{1,2,\ldots,n\} \setminus \{i,j\}} a_{i,s}.(a_{s,j})^{-1} = 0$, *for all* $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$.

We shall call such a matrix is reciprocal, if also

4. $a_{i,j} = (a_{j,i})^{-1}$, *for all* $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$.

If matrix $[C]_m$ is Jacket Conference matrix, then we define matrix $[C]'_m$ as follow

$$[C]'_m : c'_{ij} = \begin{cases} (c_{ji})^{-1}, & i \neq j; \\ 0, & i = j. \end{cases}$$

It is easy to check, that the calculations in previous section continue to be true if we use prime instead of transpose symbol t. It is enough to check $([A]_n \otimes [C]_m)([A]'_n \otimes [C]'_m) = (mn - n)[I]_{mn}$.

If we try to construct a reciprocal Jacket Conference matrix $[JC]$ of order 4:

$$[JC]_4 = \begin{bmatrix} 0 & a & b & c \\ 1/a & 0 & d & e \\ 1/b & 1/d & 0 & f \\ 1/c & 1/e & 1/f & 0 \end{bmatrix}_4 , \quad \text{we obtain} \quad \left| \begin{array}{l} c = -adf \\ b = \ iad \\ e = -idf \end{array} \right. .$$

Here, i is the imaginary unit. Selecting $a = d = f = 1$, and calculating the other coefficient by the formulae above, we can obtain a reciprocal Jacket Conference matrix

$$[JC]_4 = \begin{bmatrix} 0 & 1 & i & -1 \\ 1 & 0 & 1 & -i \\ -i & 1 & 0 & 1 \\ -1 & i & 1 & 0 \end{bmatrix}_4 . \tag{3}$$

Thus, applying the Paley construction we can construct larger Jacket matrices.

4 Generalized Paley construction

We proved in {Section 2}/{Section 3} that if $([A]_n, [B]_n)$ are matched, and $[C]_n$ is {symmetric}/{reciprocal Jacket} Conference matrix of order m, then the matrix $[S]_{mn}$, defined in (1) is {Hadamard}/{Jacket}.

In this section we shall find several matched pairs of matrices.

Proposition 2. *If $([A]_n, [B]_n)$ is matched pair, and $[D]_r$ is Hadamard matrix, then $([A]_n, [B]_n) \otimes [D]_r := ([A]_n \otimes [D]_r, [B]_n \otimes [D]_r)$ is a matched pair, too.*

Proof. Since $[A]_n$, $[B]_n$ and $[D]_r$ are Hadamard, the matrices $[A]_n \otimes [D]_r$ and $[B]_n \otimes [D]_r$ are also Hadamard matrices. We can obtain the following calculation to prove $([A]_n \otimes [D]_r, [B]_n \otimes [D]_r)$ is matched pair

$$[A \otimes D]_{nr}[B \otimes D]'_{nr} + [B \otimes D]_{nr}[A \otimes D]'_{nr} =$$

$$= ([A]_n[B]'_n + [B]_n[A]'_n) \otimes ([D]_r[D]'_r) = 0.$$

■

Proposition 3. *If $[X]$ and $[Y]$ are $2n \times n$ matrices and the composite matrix $\begin{bmatrix} X \\ Y \end{bmatrix}_{2n}$ is Hadamard, then*

$$([U]_{2n}, [V]_{2n}) := (\begin{bmatrix} X \\ Y \end{bmatrix}_{2n}, \begin{bmatrix} Y \\ -X \end{bmatrix}_{2n})$$

is matched pair of matrices.

Proof. The first matrix is Hadamard by definition, and obviously the second is also Hadamard. We shall check the condition (2), and write it as

$$\langle a_v, b_u \rangle + \langle a_u, b_v \rangle = 0, \forall u, v \in \{1, 2, \ldots, m\} \tag{4}$$

The matrices $[U]_{2n}$ and $[V]_{2n}$ (up to the sign) have rows of one and the same Hadamard matrix, it is easy to show, that the inner products in (4) will be 0, excluding the case in which row corresponds to its identical row. By definition

$$u_i = -v_{i+n}, \text{ for } i = 1, 2, \ldots, n$$

and

$$u_i = v_{i-n}, \text{ for } i = n+1, n+2, \ldots, 2n.$$

Thus if the first inner product in (4) is nonzero, it must be $\pm 2n$, while the other one would be $\mp 2n$. Thus the sum is 0. ■

5 Example

Applying the modified Paley transformation using matrix (3) we obtain the following Jacket matrix

$$[J]_8 = \begin{bmatrix} 1 & -1 & -i & 1 & 1 & 1 & i & -1 \\ -1 & 1 & -1 & i & 1 & 1 & 1 & -i \\ i & -1 & 1 & -1 & -i & 1 & 1 & 1 \\ 1 & -i & -1 & 1 & -1 & i & 1 & 1 \\ 1 & 1 & i & -1 & -1 & 1 & i & -1 \\ 1 & 1 & 1 & -i & 1 & -1 & 1 & -i \\ -i & 1 & 1 & 1 & -i & 1 & -1 & 1 \\ -1 & i & 1 & 1 & -1 & i & 1 & -1 \end{bmatrix}_8 .$$

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