New bounds for multiple packings of Euclidean sphere

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Abstract. Using lower bounds on distance spectrum components of a code on the Euclidean sphere, we improve the known asymptotical upper bounds on the cardinality of multiple packings of the sphere by balls of smaller radii.

Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space, and \( S^{n-1}(r) \subset \mathbb{R}^n \) be the (closed) Euclidean sphere of radius \( r \) with the center in the origin. Let further \( S^{n-1}(r, \vec{a}) \) be the open ball of radius \( r \) centered in \( \vec{a} \in \mathbb{R}^n \). Multiple \( L \)-packing \( \mathcal{K}(L, t) \) by balls of radius \( t \) is a finite set (\( \equiv \)code) \( \mathcal{K} \subset S^{n-1}(1) \), such that for any subset \( \{\bar{x}_1, \ldots, \bar{x}_{L+1}\} \subset \mathcal{K} \) of \( L + 1 \) points (\( \equiv \)codewords) we have

\[
\bigcap_{i=1}^{L+1} S^n(t, \bar{x}_i) = \emptyset.
\]

In other words, any point on the unit sphere can be at distance not exceeding \( t \) from at most \( L \) points from \( \mathcal{K} \).

Let \( R(\mathcal{K}(L, t)) = \frac{\ln|\mathcal{K}(L, t)|}{n} \) be the rate of the multiple packing. The problem is to find bounds on the value

\[
R(L)(t) = \limsup_{n \to \infty} \max R(\mathcal{K}(L, t)).
\]

The value \( R(L)(t) \) has been studied before, e.g. in connection with list decoding in Gaussian channel, see [1, 2] and references therein. The best known bounds are as follows.

Theorem 1

\[
R(L)(t) \geq \frac{1}{2} \ln \frac{L}{(L+1)t^2} + \frac{1}{2L} \ln \frac{1}{(L+1)(1-t^2)}.
\]

\[\text{(1)}\]

\[
R(L)(t) \leq \frac{1}{2} \ln \frac{L}{(L+1)t^2}.
\]

\[\text{(2)}\]

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Bound (1) was proved in [1], while bound (2) was first proved in [2] and later in [1] using an essentially different approach. In this work we present further improvement of the upper bound (2).

Throughout we identify a point in $\mathbb{R}^n$ with the vector from the origin to the point. Denote
\[
\alpha(\theta) = 2 \arcsin \frac{\sin(\theta/2)}{\sin \varphi}, \quad \beta(\theta) = \arccos \frac{\cos \varphi}{\cos(\theta/2)},
\]
\[
j(x, y) = (1 + y)H \left( \frac{y}{y + 1} \right) - \ln \left( \frac{1}{2} (x + \sqrt{(1 + 2y)^2x^2 - 4y(1 + y)}) \right)
\]
\[+ (1 + 2y) \ln \left( \frac{(1 + 2y)x + \sqrt{(1 + 2y)^2x^2 - 4y(1 + y)}}{2(1 + y)} \right),
\]
where $H(z) = -z \ln z - (1 - z) \ln (1 - z)$, $z \in [0, 1]$. For a given $R$ denote by $\rho_L$ the unique solution of
\[
R = (1 + \rho)H \left( \frac{\rho}{1 + \rho} \right),
\]
and denote by $\theta_L$ the unique solution of
\[
R = R(\theta_L) = \frac{1 + \sin \theta_L}{2 \sin \theta_L} H \left( \frac{1 - \sin \theta_L}{1 + \sin \theta_L} \right).
\]

Note that
\[
\frac{2\sqrt{\rho_L(\rho_L + 1)}}{1 + 2\rho_L} = \cos \theta_L,
\]
and if $\theta$ is the minimal angle between a pair of points from $\mathcal{K} \subset S^{n-1}(1)$, the rate of this code satisfies [3]:
\[
R \leq R(\theta). \tag{3}
\]

Denote also
\[
b_\mathcal{K}(x, \bar{c}) = \left\{ \bar{c}_1 : (\bar{c}, \bar{c}_1) \in \mathcal{K} \times \mathcal{K} : (\bar{c}, \bar{c}_1)/(||\bar{c}|| \cdot ||\bar{c}_1||) = x \right\},
\]
where $(\bar{a}, \bar{b}) = a_1b_1 + \ldots + a_nb_n$ stands for the scalar product. We will use the following result from [4].

**Theorem 2** For $\mathcal{K} \subset S^{n-1}(r)$, with $\ln |\mathcal{K}| = Rn(1 + o(1))$, and $\rho$ and $\varphi$ satisfying
\[
0 \leq \rho \leq \rho_L(R + \sin \varphi), \quad e^{-R} \leq \varphi \leq \pi/2,
\]
there exists $\theta$, and $\bar{c} \in \mathcal{K}$, such that
\[
2\sqrt{\rho(1 + \rho)/(1 + 2\rho)} \leq \cos \alpha(\theta) \leq 1
\]
and
\[
\frac{1}{n} b_{\mathcal{K}}(\cos \theta, \bar{c}) \geq R + 2 \ln \sin \varphi + \ln \sin \beta_{\varphi}(\theta) - j(\cos \alpha_{\varphi}(\theta), \rho).
\]

It is easy to see that \( b_{\mathcal{K}}(\cos \theta, \bar{c}) \) is the number of codewords in the cross-section of the unit sphere by the hyperplane orthogonal to the vector \( \bar{c} \), and intersecting \( \bar{c} \) in the point \( r \cdot \cos \theta \cdot \bar{c}/\|\bar{c}\| \).

Now we describe a recursive procedure of constructing a simplex of \( L + 1 \) codewords having sufficiently small pairwise distances. We start with a code \( \mathcal{K} \subset S^{n-1}(1) \) and consider the cross-section of \( S^{n-1}(1) \) by the hyperplane. Then for each \( \varphi \) and \( \rho \), as in Theorem 2, there exists \( \theta \), as in Theorem 2, such that for the cross-section of \( S^{n-1}(1) \) by the hyperplane which is orthogonal to some vector \( \bar{c} \) and being at distance \( \cos \theta \) from origin, \( b_{\mathcal{K}}(\cos \theta, \bar{c}) \) satisfies (4).

Note that the cross-section of the sphere by hyperplane is again a sphere of dimension \( n - 2 \), having radius \( r \cdot \sin \theta \) and centered in \( r \cdot \cos \theta \cdot \bar{c}/\|\bar{c}\| \). Next we shift the center of this new sphere to the origin, and once again consider the cross-section of this sphere by the hyperplane as in the previous step. Now, the new code \( \mathcal{K}_1 \) has at least \( b_{\mathcal{K}}(\cos \theta, \bar{c}) \) codewords and its cross-section contains at least \( b_{\mathcal{K}_1}(\cos \theta_1, \bar{c}_1) \) codewords. Notice that on the second step we choose new \( \varphi_1 \) and \( \rho_1 \). The procedure can be continued.

Let us provide a formal description of the procedure. On the 0-th step we have a code \( \mathcal{K}_0 \subset S^{n-1}(1) \), \( \ln |\mathcal{K}_0| \sim [nR] \). On the \( i \)-th step, \( i \geq 1 \), we obtain a code \( \mathcal{K}_i \) of rate
\[
\ln |\mathcal{K}_i|/n \sim R_i \geq R_{i-1} + 2 \ln \sin \varphi_i - \ln \beta_{\varphi_i}(\theta_i) - j(\cos \alpha_{\varphi_i}(\theta_i), \rho_i).
\]

We implement this action \( L + 1 \) times, and on the \( i \)-th step, \( i \geq 1 \), we find (if \( R_i > 0 \)) a new codeword \( \bar{c}_i \) such that its distance from \( \bar{c}_j \), \( 0 \leq j < i \), is \( d_j = 2r_j \cdot \sin \theta_j/2 \). We stop when we fix \( L + 1 \) codewords \( \bar{c}_i \in \mathcal{K}_i \subset \mathcal{K} \), \( i = 0, 1, \ldots, L \). Note also, that \( \mathcal{K}_L \subset \mathcal{K}_{L-1} \subset \cdots \subset \mathcal{K}_0 \). What should be done next, is to optimize the set \( d_j, j = 0, 1, \ldots, L \), in such a way that the simplex \( \bar{c}_i \), \( i = 0, 1, \ldots, L \), to be contained in a closed ball of the minimum possible radius \( t \). This means that there exists a point on \( S^{n-1}(1) \) which is covered by \( L + 1 \) balls, which yields that an arbitrary code of rate \( R \) on the Euclidean sphere is \( L \)-packing by the balls of radius strictly less than \( t \). Note that it is necessary to optimize over \( \varphi_i \) and \( \rho_i \) in such a way that \( R_i > 0 \). It is unlikely that this can be done analytically, however this is an affordable task for the computer. One can easily derive the expression for \( t \) as a function of pairwise distances \( d_{ij} = d_j = \|\bar{c}_i - \bar{c}_j\|, j < i \), see e.g. [5].

Consider the simplest case of \( L = 2 \). Let us have a code \( \mathcal{K}_0 \subset S^{n-1}(1) \) of rate \( R_0 \). Set \( \varphi_0 = \pi/2, \rho_0 = \rho_L - \varepsilon \), for some \( \varepsilon \ll \rho_L \). The function \( j(x, \rho) \) is increasing with \( x \in [0, 1] \),
\[
j(1, \rho) = (1 + \rho) H \left( \frac{\rho}{\rho + 1} \right).
\]
Then for the rate $R_1$ and some $\theta_0 < \theta_L + \delta$,

$$R_1 \geq R - (1 + \rho_L)H\left(\frac{\rho_L}{\rho_L + 1}\right) + \epsilon_1 = \epsilon_1.$$ 

Next, for the code $K_1 \subset S^{n-2}(\sin \theta_0)$ of rate $R_1$ we choose $\theta_1 = \pi/2$. We have $d_0 = 2\sin(\theta_0/2)$, $d_1 = \sqrt{2}\sin \theta_0$. The points $\bar{c}_0$, $\bar{c}_1$ and $\bar{c}_2$ are the vertices of a triangle with edges $d_0$, $d_0$ and $d_1$. The minimal radius $t$ of the circle passing through these vertices is

$$t = \frac{d_0}{2\sqrt{1 - \frac{d_2^2}{d_0^2}}} = \frac{\sqrt{2}\sin(\theta_0/2)}{\sqrt{1 + \sin^2(\theta_0/2)}.}$$

Then

$$\sin \theta_0 = \frac{2\sqrt{2}t\sqrt{1 - t^2}}{2 - t^2}.$$ 

Since $\theta_0 \sim \theta$ from (3), we obtain

$$R_2(t) \leq \frac{2 - t^2 + 2\sqrt{2}\sqrt{1 - t^2}}{4\sqrt{2}\sqrt{1 - t^2}}H\left(\frac{1 - 2\sqrt{2}\sqrt{1 - t^2}}{2 - \sqrt{2}\sqrt{1 - t^2}}\right) - \frac{2\sqrt{2}\sqrt{1 - t^2}}{2 - t^2}.$$ 

Comparing it to the specification of (2) for $L = 2$,

$$R_2(t) \leq \frac{1}{2} \ln \frac{2}{3t^2},$$

we conclude that (4) is tighter.

**References**


