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New bounds for multiple packings of Euclidean sphere

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Abstract. Using lower bounds on distance spectrum components of a code on the Euclidean sphere, we improve the known asymptotical upper bounds on the cardinality of multiple packings of the sphere by balls of smaller radii.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and $S^{n-1}(r) \subset \mathbb{R}^n$ be the (closed) Euclidean sphere of radius r with the center in the origin. Let further $\tilde{S}^{n-1}(r, \bar{a})$ be the open ball of radius r centered in $\bar{a} \in \mathbb{R}^n$. Multiple L-packing $\mathcal{K}(L,t)$ by balls of radius t is a finite set ($\equiv$code) $\mathcal{K} \subset S^{n-1}(1)$, such that for any subset $\{\bar{x}_1, \ldots, \bar{x}_{L+1}\} \subset \mathcal{K}$ of $L+1$ points ($\equiv$codewords) we have

$$\bigcap_{i=1}^{L+1} \tilde{S}^n(t, \bar{x}_i) = \emptyset.$$

In other words, any point on the unit sphere can be at distance not exceeding t from at most L points from $\mathcal{K}$.

Let $R(\mathcal{K}(L,t)) = \frac{\ln |\mathcal{K}(L,t)|}{n}$ be the rate of the multiple packing. The problem is to find bounds on the value

$$R_L(t) = \limsup_{n\to\infty} \max R(\mathcal{K}(L,t)).$$

The value $R_L(t)$ has been studied before, e.g. in connection with list decoding in Gaussian channel, see [1, 2] and references therein. The best known bounds are as follows.

Theorem 1

$$R_L(t) \geq \frac{1}{2} \ln \frac{L}{(L+1)t^2} + \frac{1}{2L} \ln \frac{1}{(L+1)(1-t^2)}. \tag{1}$$

$$R_L(t) \leq \frac{1}{2} \ln \frac{L}{(L+1)t^2}. \tag{2}$$

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Bound (1) was proved in [1], while bound (2) was first proved in [2] and later in [1] using an essentially different approach. In this work we present further improvement of the upper bound (2).

Throughout we identify a point in $\mathbb{R}^n$ with the vector from the origin to the point. Denote

$$\begin{aligned} \alpha_\varphi(\theta) &= 2\arcsin\frac{\sin(\theta/2)}{\sin\varphi}, \quad \beta_\varphi(\theta) = \arccos\frac{\cos\varphi}{\cos(\theta/2)}, \\ j(x,y) &= (1+y)H\left(\frac{y}{y+1}\right) - \ln\left(\frac{1}{2}(x+\sqrt{(1+2y)^2x^2-4y(1+y)}\right) \\ &\quad +(1+2y)\ln\frac{(1+2y)x+\sqrt{(1+2y)^2x^2-4y(1+y)}}{2(1+y)}, \end{aligned}$$

where $H(z) = -z\ln z - (1-z)\ln(1-z),\ z\in[0,1]$. For a given R denote by ρ_L the unique solution of

$$R = (1+\rho)H\left(\frac{\rho}{1+\rho}\right),$$

and denote by θ_L the unique solution of

$$R = R(\theta_L) = \frac{1+\sin\theta_L}{2\sin\theta_L}H\left(\frac{1-\sin\theta_L}{1+\sin\theta_L}\right).$$

Note that

$$\frac{2\sqrt{\rho_L(\rho_L+1)}}{1+2\rho_L} = \cos\theta_L,$$

and if θ is the minimal angle between a pair of points from $\mathcal{K}\subset S^{n-1}(1)$, the rate of this code satisfies [3]:

$$R \le R(\theta). \tag{3}$$

Denote also

$$b_{\mathcal{K}}(x,\bar{c}) = \left|\left\{\bar{c}_1 :\ (\bar{c},\bar{c}_1)\in\mathcal{K}\times\mathcal{K} :\ (\bar{c},\bar{c}_1)/(\|\bar{c}\|\cdot||\bar{c}_1||) = x\right\}\right|,$$

where $(\bar{a},\bar{b}) = a_1b_1+\ldots+a_nb_n$ stands for the scalar product. We will use the following result from [4].

Theorem 2 *For* $\mathcal{K}\subset S^{n-1}(r)$, *with* $\ln|\mathcal{K}| = Rn(1+o(1))$, *and* ρ *and* ϕ *satisfying*

$$0\le\rho\le\rho_L(R+\sin\varphi), \quad e^{-R}\le\varphi\le\pi/2,$$

there exists θ, *and* $\bar{c}\in\mathcal{K}$, *such that*

$$2\sqrt{\rho(1+\rho)}/(1+2\rho) \le \cos\alpha_\varphi(\theta) \le 1$$

and

$$\frac{1}{n}b_{\mathcal{K}}(\cos\theta,\bar{c}) \geq R + 2\ln\sin\varphi + \ln\sin\beta_{\varphi}(\theta) - j(\cos\alpha_{\varphi}(\theta),\rho).$$

It is easy to see that $b_{\mathcal{K}}(\cos\theta,\bar{c})$ is the number of codewords in the cross-section of the unit sphere by the hyperplane orthogonal to the vector $\bar{c}$, and intersecting $\bar{c}$ in the point $r\cdot\cos\theta\cdot\bar{c}/\|\bar{c}\|$.

Now we describe a recursive procedure of constructing a simplex of $L+1$ codewords having sufficiently small pairwise distances. We start with a code $\mathcal{K}\subset S^{n-1}(1)$ and consider the cross-section of $S^{n-1}(1)$ by the hyperplane. Then for each φ and ρ, as in Theorem 2, there exists θ, as in Theorem 2, such that for the cross-section of $S^{n-1}(1)$ by the hyperplane which is orthogonal to some vector $\bar{c}$ and being at distance $\cos\theta$ from origin, $b_{\mathcal{K}}(\cos\theta,\bar{c})$ satisfies (4). Note that the cross-section of the sphere by hyperplane is again a sphere of dimension $n-2$, having radius $r\cdot\sin\theta$ and centered in $r\cdot\cos\cdot\theta\cdot\bar{c}/\|\bar{c}\|$. Next we shift the center of this new sphere to the origin, and once again consider the cross-section of this sphere by the hyperplane as in the previous step. Now, the new code $\mathcal{K}_1$ has at least $b_{\mathcal{K}}(\cos\theta,\bar{c})$ codewords and its cross-section contains at least $b_{\mathcal{K}_1}(\cos\theta_1,\bar{c}_1)$ codewords. Notice that on the second step we choose new φ_1 and ρ_1. The procedure can be continued.

Let us provide a formal description of the procedure. On the 0-th step we have a code $\mathcal{K}_0\subset S^{n-1}(1)$, $\ln|\mathcal{K}_0|\sim[nR]$. On $i-$th step, $i\geq 1$, we obtain a code $\mathcal{K}_i$ of rate

$$\ln|\mathcal{K}_i|/n \sim R_i \geq R_{i-1} + 2\ln\sin\varphi_i - \ln\beta_{\varphi_i}(\theta_i) - j(\cos\alpha_{\varphi_i}(\theta_i),\rho_i).$$

We implement this action $L+1$ times, and on the i-th step, $i\geq 1$, we find (if $R_i>0$) a new codeword $\bar{c}_i$ such that its distance from $\bar{c}_j$, $0\leq j<i$, is $d_j = 2r_j\cdot\sin\theta_j/2$. We stop when we fix $L+1$ codewords $\bar{c}_i\in\mathcal{K}_i\subset\mathcal{K}$, $i=0,1,\ldots,L$. Note also, that $\mathcal{K}_L\subset\mathcal{K}_{L-1}\subset\cdots\subset\mathcal{K}_0$. What should be done next, is to optimize the set d_j, $j=0,1,\ldots,L$, in such a way that the simplex $\bar{c}_i$, $i=0,1,\ldots,L$, to be contained in a closed ball of the minimum possible radius t. This means that there exists a point on $S^{n-1}(1)$ which is covered by $L+1$ balls, which yields that an arbitrary code of rate R on the Euclidean sphere is L-packing by the balls of radius strictly less than t. Note that it is necessary to optimize over φ_i and ρ_i in such a way that $R_i>0$. It is unlikely that this can be done analytically, however this is an affordable task for the computer. One can easily derive the expression for t as a function of pairwise distances $d_{ij} = d_j = \|\bar{c}_i - \bar{c}_j\|, j<i$, see e.g. [5].

Consider the simplest case of $L=2$. Let us have a code $\mathcal{K}_0\subset S^{n-1}(1)$ of rate R_0. Set $\varphi_0 = \pi/2$, $\rho_0 = \rho_L - \varepsilon$, for some $\varepsilon \ll \rho_L$. The function $j(x,\rho)$ is increasing with $x\in[0,1]$,

$$j(1,\rho) = (1+\rho)H\left(\frac{\rho}{\rho+1}\right).$$

Then for the rate R_1 and some $\theta_0 < \theta_L + \delta$,

$$R_1 \geq R - (1+\rho_L)H\left(\frac{\rho_L}{\rho_L+1}\right) + \epsilon_1 = \epsilon_1.$$

Next, for the code $\mathcal{K}_1 \subset S^{n-2}(\sin\theta_0)$ of rate R_1 we choose $\theta_1 = \pi/2$. We have $d_0 = 2\sin(\theta_0/2),\ d_1 = \sqrt{2}\sin\theta_0$. The points $\bar{c}_0$, $\bar{c}_1$ and $\bar{c}_2$ are the vertices of a triangle with edges d_0, d_0 and d_1. The minimal radius t of the circle passing through these vertices is

$$t = \frac{d_0}{2\sqrt{1-\frac{d_1^2}{4d_0^2}}} = \frac{\sqrt{2}\sin(\theta_0/2)}{\sqrt{1+\sin^2(\theta_0/2)}}.$$

Then

$$\sin\theta_0 = \frac{2\sqrt{2}t\sqrt{1-t^2}}{2-t^2}.$$

Since $\theta_0 \sim \theta$ from (3), we obtain

$$R_2(t) \leq \frac{2-t^2+2\sqrt{2}\sqrt{1-t^2}}{4\sqrt{2}\sqrt{1-t^2}} H\left(\frac{1-\frac{2\sqrt{2}\sqrt{1-t^2}}{2-t^2}}{1+\frac{2\sqrt{2}\sqrt{1-t^2}}{2-t^2}}\right). \quad (4)$$

Comparing it to the specification of (2) for $L=2$,

$$R_2(t) \leq \frac{1}{2}\ln\frac{2}{3t^2},$$

we conclude that (4) is tighter.

References

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