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A note on a result by Hamada on minihypers

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Abstract. Hamada [Bull. Osaka Women's Univ. 24:1–47, 1985; Discrete Math. 116:229-268, 1993] characterized the non-weighted minihypers having parameters $(\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; t, q)$ with $t > \lambda_1 > \lambda_2 > \cdots > \lambda_h \geq 0$. This result has been generalized in [Des. Codes Cryptogr. 45:123-138,2007] where it was proved that a weighted $(\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; t, q)$-minihyper $\mathfrak{F}$, with $k-1 > \lambda_1 > \lambda_2 > \cdots > \lambda_h \geq 0$, is a sum of the characteristic functions of spaces of dimension $\lambda_1, \ldots, \lambda_h$. In this note, we prove that we can relax further the restrictions on the integers λ_i by allowing $r(q) - 1$ equalities in the chain of strict inequalities $\lambda_2 > \ldots > \lambda_h$.

1 Introduction

Let $\mathrm{PG}(t,q)$ be the t-dimensional projective space over $\mathbb{F}_q$. Denote by $\mathcal{P}$ the set of points of the projective geometry $\mathrm{PG}(t,q)$ and let $v_{t+1} = (q^{t+1}-1)/(q-1)$ denote the cardinality of $\mathcal{P}$. A *multiset* in $\mathrm{PG}(t,q)$ is any mapping $\mathfrak{K}\colon \mathcal{P} \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all nonnegative integers. This mapping is extended in a natural way to the subsets of $\mathcal{P}$ (the extension is also denoted by $\mathfrak{K}$) by $\mathfrak{K}(\mathcal{Q}) = \sum_{P \in \mathcal{Q}} \mathfrak{K}(P)$ where $\mathcal{Q} \subseteq \mathcal{P}$. The integer $\mathfrak{K}(\mathcal{Q})$ is called the *multiplicity* of $\mathcal{Q}$. The *cardinality of a multiset* is defined by $|\mathfrak{K}| = \mathfrak{K}(\mathcal{P})$. The *support* $\mathrm{supp}\,\mathfrak{K}$ of a multiset $\mathfrak{K}$ is defined as $\mathrm{supp}\,\mathfrak{K} = \{P \in \mathcal{P} \mid \mathfrak{K}(P) > 0\}$. A multiset with $\mathfrak{K}(P) \in \{0,1\}$ for every $P \in \mathcal{P}$ is called a *non-weighted* or *projective* multiset. Projective multisets can be viewed as sets by identifying them with their support.

Let $\mathcal{Q}$ be a set of points in $\mathrm{PG}(t,q)$. We define the characteristic multiset $\chi_{\mathcal{Q}}$ by

$$\chi_{\mathcal{Q}}(P) = \begin{cases} 1 & \text{if } P \in \mathcal{Q}, \\ 0 & \text{if } P \notin \mathcal{Q}. \end{cases}$$

Definition 1. *A multiset $\mathfrak{F}$ in $\mathrm{PG}(t,q)$, $t \geq 2$, is called an $(f,m;t,q)$*-minihyper *or (f,m)-minihyper if* *(a)* $\mathfrak{F}(\mathcal{P}) = f$;

(b) $\mathfrak{F}(H) \geq m$ *for any hyperplane* H;

(c) there exists a hyperplane H_0 *with* $\mathfrak{F}(H_0) = m$.

This definition of a minihyper is equivalent to the original one given by Hamada and Tamari in [3]. In order to save space, we refer to [4] for all notions that are not defined here.

In [1] and [2], Hamada characterized the non-weighted minihypers with parameters $(\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; t, q)$ with $t > \lambda_1 > \lambda_2 > \cdots > \lambda_h \geq 0$, as the union of a λ_1-dimensional space, λ_2-dimensional space, ..., λ_h-dimensional space, which all are pairwise disjoint. In [4], this result was extended to weighted minihypers.

Theorem 1. *Let* $\mathfrak{F}$ *be a* $(\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; t, q)$*-minihyper, with* $t \geq 2$*,* $q \geq 3$*, and*

$$t > \lambda_1 > \lambda_2 > \ldots > \lambda_h \geq 0.$$

Then

$$\mathfrak{F} = \sum_{i=1}^{h} \chi_{\pi_i},$$

where π_i *is a* λ_i*-dimensional subspace of* $\mathrm{PG}(t, q)$*,* $i = 1, \ldots, h$.

In this note, we show that we can relax the restrictions on the numbers λ_i by allowing some of them to be equal. We prove the following theorem.

Theorem 2. *Let* $t \geq 2$ *be an integer and let* $q \geq 3$ *be a prime power. Let* $\lambda_1, \ldots, \lambda_h$ *be a sequence of non-negative integers such that*

(1) $t > \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_h \geq 0$*, and*

(2) equalities in (1) occur in at most $r(q) - 1$ *places, where* $q + 1 + r(q)$ *is the size of the smallest nontrivial blocking set in* $\mathrm{PG}(2, q)$.

Then every minihyper $\mathfrak{F}$ *in* $\mathrm{PG}(t, q)$ *with parameters* $(\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i})$ *can be represented as*

$$\mathfrak{F} = \sum_{i=1}^{h} \chi_{\pi_i},$$

where π_i *is a* λ_i*-dimensional subspace of* $\mathrm{PG}(t, q)$*,* $i = 1, \ldots, h$.

2 The proof of Theorem 2

For the proof of our characterization result, we use induction on t and h. The theorem is obviously true for $h = 1$ for every t and for $t = 2$ for all $h \leq t + r(q) - 1$. Note that for $t = 2$, our statement follows by the definition of $r(q)$, namely, every blocking set with less than $q + r(q) + 1$ points contains a line in its support.

Lemma 3. *Let the integers $t, q, \lambda_1, \ldots, \lambda_h$ satisfy the conditions of Theorem 2. Let further $\mathfrak{F}$ be a $(\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; t, q)$-minihyper. Then* $\operatorname{supp}\mathfrak{F}$ *contains a λ_1-dimensional subspace of* $\mathrm{PG}(t, q)$.

The proof of this fact follows *mutatis mutandis* the proof of Lemma 17 from [4].

Now we assume that Theorem 2 is proved for all dimensions up to $t_0 - 1$ for all possible h, and for dimension t_0 for all $(\sum_{i=1}^{h'} v_{\lambda_i+1}, \sum_{i=1}^{h'} v_{\lambda_i})$-minihypers with $t > \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda'_h \geq 0$, where $h' < h_0$. We want to prove that theorem for minihypers for which the sums in the parameters contain h_0 summands.

An easy counting argument shows that for a subspace S of codimension s

$$\mathfrak{F}(S) \geq v_{\lambda_1 - s + 1} + \ldots + v_{\lambda_{h_0} - s + 1}.$$

Here, $v_\alpha = 0$ for $\alpha < 0$. Hence the minimal multiplicity of a subspace S of dimension $t_0 - 2$ (codimension 2) is $v_{\lambda_1 - 1} + \ldots + v_{\lambda_{h_0} - 1}$ and all hyperplanes through S are also of minimal multiplicity $v_{\lambda_1} + \ldots + v_{\lambda_{h_0}}$.

(1) Assume that $\lambda_1 < t_0 - 1$. Then each hyperplane contains a 0-point. Consider a projection φ from an arbitrary 0-point P onto a hyperplane Δ. The induced minihyper $\mathfrak{F}^\varphi$ in $\Delta \cong \mathrm{PG}(t_0 - 1, q)$ has the parameters of $\mathfrak{F}$: $(\sum_{i=1}^{h_0} v_{\lambda_{i+1}}, \sum_{i=1}^{h_0} v_{\lambda_i})$. By the induction hypothesis

$$\mathfrak{F}^\varphi = \chi_{\delta_1} + \chi_{\delta_2} + \ldots + \chi_{\delta_{h_0}}, \tag{1}$$

where δ_i is a subspace of Δ with $\dim \delta_i = \lambda_i$, $i = 1, \ldots, h_0$.

The support of $\mathfrak{F}$ contains a λ_1-dimensional subspace π_1 (cf. Lemma 3). Set $\mathfrak{F}' = \mathfrak{F} - \chi_{\pi_1}$. Let H be a hyperplane that contains π_1. Consider a projection from a 0-point P in H. Clearly, $\mathfrak{F}^\varphi$ has the form (1). The image of π_1 under φ is exactly δ_1. Thus

$$\mathfrak{F}'^\varphi = \chi_{\delta_2} + \ldots + \chi_{\delta_{h_0}}.$$

This implies that

$$\mathfrak{F}'(H) = \mathfrak{F}'^\varphi(H') = v_{\lambda_2} + \ldots + v_{\lambda_{h_0}}.$$

Let H be a hyperplane that does not contain π_1. Now H meets π_1 in a $(\lambda_1 - 1)$-dimensional subspace and again

$$\mathfrak{F}'(H) = \mathfrak{F}(H) - |\pi_1 \cap H| = \mathfrak{F}(H) - v_{\lambda_1} \geq v_{\lambda_2} + \ldots + v_{\lambda_{h_0}}.$$

Hence $\mathfrak{F}'$ is a $(\sum_{i=2}^{h_0} v_{\lambda_{i+1}}, \sum_{i=2}^{h_0} v_{\lambda_i})$-minihyper in $\mathrm{PG}(t_0, q)$. By the induction hypothesis, $\mathfrak{F}' = \chi_{\pi_2} + \ldots + \chi_{\pi_{h_0}}$, where dim π_i are λ_i-dimensional subspaces of $\mathrm{PG}(t_0, q)$, $i = 2, \ldots, h_0$.

(2) Let $\lambda_1 = t_0 - 1$. Again by Lemma 3, we have that $\operatorname{supp} \mathfrak{F}$ contains a λ_1-space, i.e. a hyperplane π_1. Define the multiset $\mathfrak{F}' = \mathfrak{F} - \chi_{\pi_1}$. For each hyperplane $H \neq \pi_1$ we have $\mathfrak{F}'(H) \geq v_{\lambda_2} + \ldots + v_{\lambda_h}$. Now it is enough to demonstrate that

$$\mathfrak{F}'(\pi_1) \geq v_{\lambda_2} + \ldots + v_{\lambda_h}.$$

This will imply that $\mathfrak{F}'$ is a $(\sum_{i=2}^{h} v_{\lambda_i+1}, \sum_{i=2}^{h} v_{\lambda_i})$-minihyper and the result will follow by induction.

Fix a $(t_0 - 2)$-dimensional subspace δ of $\mathrm{PG}(t_0, q)$ of minimal multiplicity. Denote by Π_i, $i = 0, \ldots, q$, the hyperplanes through δ. All these hyperplanes are also of minimal multiplicity: $\mathfrak{F}(\Pi_j) = \sum_{i=1}^{h} v_{\lambda_i}$. By the induction hypothesis, the restriction of $\mathfrak{F}$ to every Π_j is a sum of subspaces:

$$\mathfrak{F}|_{\Pi_j} = \chi_{\pi_1^{(j)}} + \chi_{\pi_2^{(j)}} + \ldots + \chi_{\pi_h^{(j)}},$$

where $\dim \pi_i^{(j)} = \lambda_i - 1$, $i = 1, \ldots, h_0$, $j = 0, \ldots, q$. Note that the indices $i \in \{1, \ldots, h_0\}$ can be chosen in such way that the subspaces $\pi_i^{(j)}$ meet δ in the same $(\lambda_i - 2)$-dimensional subspace. This follows by the fact that δ is of minimal multiplicity. In other words, we can arrange the subspaces $\pi_i^{(j)}$ in such way that

$$\pi_i^{(0)} \cap \delta = \pi_i^{(1)} \cap \delta = \ldots = \pi_i^{(q)} \cap \delta = \delta_i, \quad i = 1, 2, \ldots, h.$$

Now we can write

$$\begin{aligned}
\mathfrak{F} &= \sum_{j=0}^{q} \mathfrak{F}|_{\Pi_j} - q\mathfrak{F}|_{\delta} \\
&= \sum_{j=0}^{q}\sum_{i=1}^{h} \chi_{\pi_i^{(j)}} - q\sum_{i=1}^{h}\chi_{\delta_i} \\
&= \sum_{i=1}^{h}\sum_{j=0}^{q} \chi_{\pi_i^{(j)}} - q\sum_{i=1}^{h}\chi_{\delta_i} \\
&= \sum_{i=1}^{h}\left(\sum_{j=0}^{q} \chi_{\pi_i^{(j)}} - q\chi_{\delta_j}\right) \\
&= \sum_{i=1}^{h} \mathfrak{G}_i,
\end{aligned}$$

where we have set $\mathfrak{G}_i = \sum_{j=0}^{q} \chi_{\pi_i^{(j)}} - q\chi_{\delta_j}$. It is known that $\mathfrak{G}_1 = \chi_{\pi_1}$ which implies that $\mathfrak{F}' = \sum_{i=2}^{h} \mathfrak{G}_i$.

Let us fix an integer $i \in \{2, \ldots, h\}$. First, we consider the case when δ_i is not contained δ_1. In this case,

$$\begin{aligned}
\mathfrak{G}_i(\pi_1) &= \sum_{j=0}^{q} |\pi_1 \cap \pi_i^{(j)}| - |\pi_1 \cap \delta_i| \\
&= (q+1)v_{\lambda_i - 1} - qv_{\lambda_i - 2} \\
&= \frac{q^{\lambda_i} - 1}{q-1} = v_{\lambda_i}.
\end{aligned}$$

Now suppose that δ_i is a subspace of δ_1. Fix a $(t-3)$-dimensional subspace δ_0 of δ that has minimal multiplicity. We have $\delta_0 \neq \delta_1$ since δ_1 is not of minimal multiplicity. Denote by $\tau_1, \ldots, \tau_q$ the $(t_0 - 2)$-dimensional subspaces of Π_0 through δ_0 other than δ. Since at most $r(q) - 1$ of the subspaces τ_i are not minimal (this happens when $\lambda_{h_0 - r(q)+2} = \ldots = \lambda_{h_0} = 1$), we can assume with no loss of generality that τ_1 is minimal, i.e. $\mathfrak{F}(\tau_1) = \sum_{i=1}^{h} v_{\lambda_i}$. If $\tau_1 \cap \pi_1^{(j)}$, $i = 2, \ldots, h$, is not contained in π_1 for some j, we can repeat the above argument to show that $\mathfrak{G}_i(\pi_1) \geq v_{\lambda_i}$.

Now assume that $\tau_1 \cap \pi_i^{(0)}$ is in π_1. Clearly, $\tau_1 \cap \pi_i^{(0)}$ does not coincide with δ_i since otherwise δ would coincide with τ_1. Hence $\pi_1^{(0}$ contains other points (apart from δ_i) that are from π_1. This implies that $\pi_1^{(0)}$ is contained completely in π_1. Now we have that

$$\mathfrak{G}_i(\pi_1) = \sum_{j=0}^{q} |\pi_1 \cap \pi_i^{(j)}| - q|\pi_1 \cap \delta_i| \geq |\pi_1 \cap \pi_i^{(0)}| = |\pi_i^{(0)}| = v_{\lambda_i}.$$

Thus we have proved that in all cases $\mathfrak{G}_i(\pi_1) \geq v_{\lambda_i}$. Now we have

$$\mathfrak{F}'(\pi_1) = \sum_{i=2}^{h} \mathfrak{G}_i(\pi_1) \geq \sum_{i=2}^{h} v_{\lambda_i},$$

which finishes the proof.

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