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The nonexistence of the (23,13,2,2) superimposed codes[1]

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Abstract. The nonexistence of (23,13,2,2) superimposed codes is proved.

1 Introduction

Definition 1 *A binary $N \times T$ matrix $C = (c_{ij})$ is called an (N, T, w, r) superimposed code (SIC) if for any pair of subsets $W, R \subset \{1, 2, \ldots, T\}$ such that $|W| = w$, $|R| = r$ and $W \bigcap R = \emptyset$, there exists a coordinate $i \in \{1, 2, \ldots, N\}$ such that $c_{ij} = 1$ for all $j \in W$ and $c_{ij} = 0$ for all $j \in R$.*

Let $N(T, w, r)$ be the minimum length N for which an (N, T, w, r) SIC exists for fixed values of T, w and r. The problem of determining the exact values of $N(T, w, r)$ is completely solved only for $w = r = 1$ [6].

The exact values of $N(T, 2, 2)$ are known only for $T \leq 12$ [1], [3], [2]:

T	4	5	6	7	8	9	10	11	12
$N(T, 2, 2)$	6	10	14	14	14	18	20	22	22

A (16,26,2,2) superimposed code is constructed in [3], hence $22 \leq N(T, 2, 2) \leq 26$ for $T = 13, 14, 15, 16$.

The main result of this article is that there is no (23,13,2,2) superimposed code. Consequently $24 \leq N(13, 2, 2) \leq 26$ for $T = 13, 14, 15, 16$.

2 Preliminaries

For a binary matrix or vector C denote by $wt(C)$ the number of 1's in C, and by $wt(\overline{C})$ the number of 0's in C.

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For a binary matrix C denote by $d(x,y)$ the Hamming distance between two columns x and y of C. Let $d_2 = \min\{d(x,y) \mid x,y \in C,\ x \neq y\}$ and $d(C) = \sum_{x,y\in C,\ x\neq y} d(x,y)$.

Lemma 2 *(Plotkin bound) [5]* $\binom{T}{2} d_2 \leq N \left\lfloor \frac{T}{2} \right\rfloor \left\lfloor \frac{T+1}{2} \right\rfloor$.

Definition 3 *Let x be a column in the superimposed code C. The residual code $Res(C, x=a)$ is the code obtained in the following way:*
1) take the i-th row $(i = 1, 2, ..., N)$ iff $c_{i,x} = a$;
2) delete the column x in the selected rows.

We will use the shorter notation $Res(C,\ x = a,\ y = b)$ instead of $Res(Res(C, x=a), y=b)$.

Lemma 4 *Suppose C is an (N,T,w,r) superimposed code with $w > 1,\ r > 1$. Then*
(a) $N(T-1, w-1, r) \leq wt(x) \leq N - N(T-1, w, r-1)$ for any column x;
(b) $d_2 \geq 2N(T-2, w-1, r-1)$.

Proof. (a) The residual code $Res(C, x=1)$ is a $(wt(x), T-1, w-1, r)$ SIC, while $Res(C, x=0)$ is a $(wt(\overline{x}), T-1, w, r-1)$ SIC. Hence $wt(x) \geq N(T-1, w-1, r)$ and $wt(\overline{x}) \geq N(T-1, w, r-1)$.
(b) Let x and y be an arbitrary pair of different columns of C. The residual codes $Res(C, x=1, y=0)$ and $Res(C, x=0, y=1)$ are $(N', T-2, w-1, r-1)$ and $(N'', T-2, w-1, r-1)$ SIC, respectively. Hence $d(x,y) = N' + N'' \geq 2N(T-2, w-1, r-1)$. □

Lemma 5 *Let x be a column of C and $A = Res(C, x=1)$. Then*

$$d(A) + wt(\overline{A}) \geq \binom{T}{2} d_2 - (N - wt(x)) \left\lfloor \frac{T}{2} \right\rfloor \left\lfloor \frac{T+1}{2} \right\rfloor$$

Proof. Denote by C_1 the submatrix of C, containing all rows with value 1 in the column x, and by C_0 the remaining part of C. Then

$$\binom{T}{2} d_2 \leq d(C) = d(C_1) + d(C_0).$$

But $d(C_1) = d(A) + wt(\overline{A})$ and $d(C_0) \leq (N - wt(x)) \left\lfloor \frac{T}{2} \right\rfloor \left\lfloor \frac{T+1}{2} \right\rfloor$.
The result follows. □

Definition 6 *Two (N,T,w,r) superimposed codes are equivalent if one of them can be obtained from the other by a permutation of the rows and a permutation of the columns. In the case $w = r$ an inversion of the all code entries is also allowed.*

3 Main result

Lemma 7 *If* C *is a* $(23, 13, 2, 2)$ *superimposed code then*
(a) $9 \leq wt(x) \leq 14$ *for any column* x *of* C*;*
(b) $d_2 = 12$.

Proof. (a) Follows from Lemma 4 and the known value $N(12, 1, 2) = 9$ [3];
(b) Follows from Lemma 2, Lemma 4 and the known value $N(11, 1, 1) = 6$ [6]. □

Theorem 8 *There is no (23,13,2,2) superimposed code.*

Proof. Suppose C is a (23,13,2,2) SIC. Up to equivalence we may assume that the code has the following form:

$$\left(\begin{array}{c|c} 1 & \\ \vdots & A \\ 1 & \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right)$$

where the matrix A is a $(N, 12, 1, 2)$ SIC where $N \in \{9, 10, 11\}$, and the matrix B has to be chosen in such a way that the whole matrix to be a $(23, 13, 2, 2)$ SIC. We may assume that the rows of B are sorted lexicographically.

Applying the method described in [4] we constructed all inequivalent $(9, 12, 1, 2)$, $(10, 12, 1, 2)$ and $(11, 12, 1, 2)$ superimposed codes. Then we checked the condition of Lemma 5, which turned out to reduce the amount of computations approximately 6 times.

SIC parameters	(9,12,1,2)	(10,12,1,2)	(11,12,1,2)
number of inequivalent SIC	1	99	243709
number of inequivalent SIC, which satisfy Lemma 5	1	54	44509

Using an exhaustive computer search we tried to construct the matrix B column by column taking into account the restrictions of Lemma 7 and the sorted rows property. It turned out, however, that superimposed codes with parameters (23,13,2,2) do not exist. □

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