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On the spectrum of sizes of complete caps in projective spaces $PG(n,q)$ of small dimension

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Dedicated to the memory of Giuseppe Pellegrino (1926-2007)

Abstract. For projective spaces $PG(n,q)$ of small dimension, new sizes of complete caps including small these are obtained. The corresponding tables are given. A generalization of Segre's construction of complete caps in $PG(3,2^h)$ is described. In $PG(2,q)$, for $q=17$, $\delta=4$, and $q=19,27$, $\delta=3$, we give complete $(\frac{1}{2}(q+3)+\delta)$-arcs other than conics and sharing $\frac{1}{2}(q+3)$ points with an irreducible conic. We have proven they are unique up to collineations.

1 Introduction

Let $PG(n,q)$ be the projective space of dimension n over the Galois field F_q of q elements. A k-cap in $PG(n,q)$ is a set of k points, no three of which are collinear. A k-cap in $PG(n,q)$ is complete if it is not contained in a $(k+1)$-cap of $PG(n,q)$. If $n=2$, then a k-cap is called a k-arc. We use the following notations for $PG(n,q)$: $m_2(n,q)$ is the size of the largest complete cap, $m_2'(n,q)$ is the size of the second largest complete cap, and $t_2(n,q)$ is the size of the smallest complete cap. The corresponding *best known* values are denoted by $\overline{m}_2(n,q)$, $\overline{m}_2'(n,q)$, and $\overline{t}_2(n,q)$.

In all tables new bounds and sizes of complete caps obtained in this work are marked by the asterisk $\star$ and are written by the bold font.

For the spectrum of possible sizes of complete caps in the spaces of small dimension, the known results are collected in [3],[6]. Using recent results from literature and computer search done in this work we obtained new upper bounds on $t_2(n,q)$ and new sizes of complete caps. As result, we essentially updated tables of [3],[6], see Tables 1-4 below.

Also we generalize Segre's construction [19] of complete caps in $PG(3,2^h)$ basing on ideas of unpublished manuscript [16], see Theorem 4.

2 Complete arcs in planes $PG(2,q)$

The smallest known sizes $\bar{t}_2 = \bar{t}_2(2,q)$ are given in Table 1. For $q = 2^7$, see [12].

Table 1: The smallest known sizes $\bar{t}_2 = \bar{t}_2(2,q) < 4\sqrt{q}$ of complete arcs in planes $PG(2,q)$. $A_q = \lfloor 4\sqrt{q} - \bar{t}_2(2,q) \rfloor$

q	$\bar{t}_2$	A_q	q	$\bar{t}_2$	A_q	q	$\bar{t}_2$	A_q	q	$\bar{t}_2$	A_q	q	$\bar{t}_2$	A_q
3	4.	2	101	30	10	256	56	8	439	78	5	641	**99**	2
4	6.	2	103	31	9	257	56	8	443	79	5	643	**99**	2
5	6.	2	107	32	9	263	**56**	8	449	80	4	647	99	2
7	6.	4	109	32	9	269	57	8	457	81	4	653	**100**	2
8	6.	5	113	33	9	271	58	7	461	81	4	659	100	2
9	6.	6	121	34	10	277	59	7	463	82	4	661	**100**	2
11	7.	6	125	35	9	281	59	8	467	82	4	673	**102**	1
13	8.	6	127	35	10	283	**59**	8	479	83	4	677	103	1
16	9.	7	128	34	11	289	60	8	487	84	4	683	103	1
17	10.	6	131	36	9	293	61	7	491	84	4	691	104	1
19	10.	7	137	37	9	307	**62**	8	499	85	4	701	**104**	1
23	10.	9	139	37	10	311	63	7	503	85	4	709	105	1
25	12.	8	149	39	9	313	63	7	509	85	5	719	106	1
27	12.	8	151	39	10	317	**63**	8	512	86	4	727	106	1
29	13.	8	157	40	10	331	**65**	7	521	**86**	5	729	104	4
31	14	8	163	41	10	337	66	7	523	**86**	5	733	107	1
32	14	8	167	42	9	343	67	7	529	88	4	739	**107**	1
37	15	9	169	**42**	10	347	67	7	541	89	4	743	**108**	1
41	16	9	173	44	8	349	**67**	7	547	**89**	4	751	108	1
43	16	10	179	44	9	353	68	7	557	**90**	4	757	109	1
47	18	9	181	45	8	359	69	6	563	92	2	761	109	1
49	18	10	191	46	9	361	69	7	569	93	2	769	110	0
53	18	11	193	47	8	367	70	6	571	93	2	773	111	0
59	20	10	197	47	9	373	71	6	577	93	3	787	112	0
61	20	11	199	47	9	379	71	6	587	94	2	797	112	0
64	22	10	211	49	9	383	71	7	593	95	2	809	113	0
67	23	9	223	51	8	389	**72**	6	599	95	2	811	**113**	0
71	22	11	227	51	9	397	73	6	601	96	2	821	114	0
73	24	10	229	52	8	401	74	6	607	**96**	2	823	**114**	0
79	26	9	233	52	9	409	75	5	613	97	2	827	**115**	0
81	26	10	239	53	8	419	76	5	617	97	2	829	**115**	0
83	27	9	241	53	9	421	**76**	6	619	**97**	2	839	**115**	0
89	28	9	243	54	8	431	77	6	625	96	4	841	112	4
97	30	9	251	55	8	433	**77**	6	631	**98**	2			

Through the paper for new computer results we used the randomized greedy algorithms [3, Sec. 2],[6, Sec. 2], the back-tracking algorithms [3, Sec. 2], the breadth-first algorithm, algorithms combining orbits of groups, and other geometrical algorithms.

Theorem 1 *In $PG(2,q)$ we have $t_2(2,q) < 4\sqrt{q}$ for $3 \le q \le 841$. In addition,*

$$\begin{array}{llll} t_2(2,q) \le 4\sqrt{q}-8 & for & 23 \le q \le 269, & q = 281, 283, 289, 307, 317; \\ t_2(2,q) \le 4\sqrt{q}-7 & for & 19 \le q \le 353, & q = 361, 383; \\ t_2(2,q) \le 4\sqrt{q}-6 & for & 9 \le q \le 401, & q = 421, 431, 433; \\ t_2(2,q) \le 4\sqrt{q}-5 & for & 8 \le q \le 443, & q = 509, 521, 523; \\ t_2(2,q) \le 4\sqrt{q}-4 & for & 7 \le q \le 557, & q = 625, 729, 841; \\ t_2(2,q) \le 4\sqrt{q}-2 & for & 3 \le q \le 661; & \\ t_2(2,q) \le 4\sqrt{q}-1 & for & 3 \le q \le 761. & \end{array}$$

Theorem 2 *There exist the following complete k-arcs in $PG(2,q)$ with $k \le m_2'(2,q)$:*
$PG(2,64): \ 22 \le k \le 35, \quad k = 42$ [5] , $k = 57$.
$PG(2,128): \ 34 \le k \le 67. \quad PG(2,163): \ 41 \le k \le 85.$
$PG(2,167): \ 42 \le k \le 87.$

Theorem 3 *In $PG(2,q)$, q odd, we denote by $K_q(\delta)$ a complete $(\frac{1}{2}(q+3)+\delta)$-arc other than conic and sharing $\frac{1}{2}(q+3)$ points with an irreducible conic. Let Δ_q be the maximal possible value of δ. Then $\Delta_{17} = 4$, $\Delta_{19} = \Delta_{27} = 3$, $\Delta_{11} = \Delta_{23} = \Delta_{29} = \Delta_{31} = 2$. There is no any arc $K_{17}(3)$. For $\delta = 3, 4$ and $q \le 27$, the arcs $K_q(\delta)$ are unique up to collineations.*

Theorem 3 is proved by an exhaustive computer search. Note also that $\Delta_{25} = 2$ [14], as $25^2 \equiv 1 \pmod{16}$, and $\Delta_{13} = 4$ [1]. One can compare the results of [17],[18] with Theorem 3 and the arcs $K_q(\Delta_q)$ written below. The unique 14-arc $K_{17}(4)$ is a counterexample to [18]. The 14-arc $K_{19}(3)$ is obtained in [18] but we have proven that it is unique. Finally, the unique 18-arc $K_{27}(3)$ is new.

Points of the unique 14-arc $K_{17}(4)$ are given in [7]: { (1,10,12),(1,6,8); (1,0,6),(1,0,11), (1,1,4),(1,1,13),(1,6,9),(1,10,5),(1,14,3),(1,3,14); (0,1,3),(0,1,0), (1,5,1),(1,14,10)}. The first ten points lie on the conic $3x_1^2 + x_2^2 = 2x_0^2$ and the last four are placed outside it. By semicolon we separate orbits of the stabilizer group. For $K_{17}(4)$ the stabilizer is the dihedral group $\mathbf{D}_4$ of order eight.

The unique 14-arc $K_{19}(3)$ obtained in this work may be represented with the following coordinates: {(1,5,6), (1,2,4); (1,0,0), (1,7,11),(1,13,17); (1,1,1),(1,3,9), (1,4,16),(1,6,17), (1,9,5),(1,17,4); (1,13,6), (1,1,11),(1,6,8)}. The first 11 points lie on the conic $x_1^2 = x_0x_2$.

The unique 18-arc $K_{27}(3)$ may be represented with the following coordinates: {(1,14,1), (1,12,23),(1,10,19); (1,0,0), (0,0,1),(1,2,3),(1,22,17),(1,13,25), (1,11,21); (1,8,15),(1,20,13), (1,19,11),(1,16,5),(1,4,7),(1,5,9); (0,1,0),(1,6,8), (1,21,12)}. The field F_{27} is generated by the polynomial $x^3 - x^2 - 2$. Elements of

F_{27} are represented as follows: $0 = 0$, $\alpha^i = i+1$, where α is a primitive element of the field. The first 15 points of $K_{27}(3)$ belong to the conic $x_1^2 = x_0x_2$.

The stabilizer group of the arcs $K_{19}(3)$ and $K_{27}(3)$ is the symmetric group $\mathbf{S}_3$ of order six. In $K_{17}(4), K_{19}(3)$, and $K_{27}(3)$ the points outside the conic lie on the same stabilizer group orbit.

3 Small caps in $PG(n,q)$, $n \geq 3$

In [19] Segre constructed complete $(3q+2)$-caps in $PG(3,2^h)$. In unpublished manuscript [16], connected with the paper [15] and cited in [13, Table 4.8], it is remarked $t_2(3,q) \leq 2q + t_2(2,q)$, $q = 2^h \geq 4$. We generalize ideas of [19],[16].

Theorem 4 *Let $q \geq 4$ be even. For every complete k_2-arc in the plane $PG(2,q)$ there is a complete $(2q+k_2)$-cap in the space $PG(3,q)$.*

Table 2 : The sizes $\bar{t}_2(n,q)$ of the known small complete caps in $PG(n,q)$

n	q	$t_2(n,q)$	$\bar{t}_2(n,q)$	new	n	q	$t_2(n,q)$	$\bar{t}_2(n,q)$	new
4	4	$19 \leq$	20	[2]	5	4	$31 \leq$	50	
4	5	$21 \leq$	31		5	5	$36 \leq$	**82**	$\star$
4	7	$29 \leq$	**56**	$\star$	5	7	$70 \leq$	**174**	$\star$
4	8	$33 \leq$	**53**	$\star$	5	8	$91 \leq$	**181**	$\star$
4	9	$39 \leq$	87		5	9	$115 \leq$	**302**	$\star$
4	11	$52 \leq$	**121**	$\star$	6	3	$34 \leq$	44	
4	13	$67 \leq$	**162**	$\star$	6	4	$61 \leq$	**114**	$\star$
4	16	$91 \leq$	153	[4]	6	5	$80 \leq$	131	
4	17	$100 \leq$	**255**	$\star$	6	7	$121 \leq$	349	
5	3	$20 \leq$	22		6	8	$256 \leq$	**437**	$\star$

Table 3 : The sizes $\bar{t}_2(3,q)$ of the known small complete caps in $PG(3,q)$

q	$t_2(3,q)$	$\bar{t}_2(3,q)$	new	q	$t_2(3,q)$	$\bar{t}_2(3,q)$	new
7	17	$3q-4=17$		43	$63 \leq$	$3q+25=$ **153**	$\star$
8	$14 \leq$	$3q-4=20$		47	$69 \leq$	$3q+28=$ **169**	$\star$
9	$15 \leq$	$3q-3=24$		49	$72 \leq$	$3q+33=$ **180**	$\star$
11	$18 \leq$	$3q-3=30$		53	$77 \leq$	$3q+36=$ **195**	$\star$
13	$21 \leq$	$3q-3=36$		59	$86 \leq$	$3q+43=$ **220**	$\star$
16	$25 \leq$	$2q+9=41$		61	$89 \leq$	$3q+47=$ **230**	$\star$
17	$26 \leq$	$3q=51$		64	$93 \leq$	$2q+22=$ **150**	Th. 4
19	$29 \leq$	$3q+1=58$		67	$97 \leq$	$3q+56=$ **257**	$\star$
23	$35 \leq$	$3q+3=72$		71	$103 \leq$	$3q+62=$ **275**	$\star$
25	$38 \leq$	$3q+6=$ **81**	$\star$	73	$106 \leq$	$3q+68=$ **287**	$\star$
27	$41 \leq$	$3q+8=$ **89**	$\star$	79	$114 \leq$	$4q-4=$ **312**	$\star$
29	$43 \leq$	$3q+9=$ **96**	$\star$	81	$117 \leq$	$4q-3=$ **321**	$\star$
31	$46 \leq$	$3q+11=$ **104**	$\star$	83	$120 \leq$	$4q-2=$ **330**	$\star$
32	$48 \leq$	$2q+14=$ **78**	Th. 4	89	$128 \leq$	$4q-1=$ **355**	$\star$
37	$55 \leq$	$3q+17=$ **128**	$\star$	97	$140 \leq$	$4q+6=$ **394**	$\star$
41	$60 \leq$	$3q+22=$ **145**	$\star$	128		$2q+34=$ **290**	[12], Th. 4

Theorem 5 *It holds that* $t_2(3,q) \leq 3q$ *if* $2 \leq q \leq 17$; $t_2(3,q) < 4q$ *if* $2 \leq q \leq 89$.

4 On the complete cap sizes in $PG(n,q)$, $n \geq 4$

Theorem 6 *The upper bound on the smallest size* $t_2(n,3)$ *of a complete cap in the ternary projective space* $PG(n,3)$ *has the form* $t_2(n,3) \leq 11 \cdot 2^{n-4}$, $n \geq 4$.

Theorem 7 *There are the following bounds*: $m_2(5,4) \leq 153$, $m_2(6,4) \leq 607$, $m_2(7,3) \leq 404$, $m_2(8,3) \leq 1208$, $m_2(9,3) \leq 3247$, $t_2(4,7) \leq 56$, $t_2(6,4) \leq 114$, $534 \leq m_2(8,3)$.

Table 4 gives sizes of the known complete caps in $PG(n,q)$, $n \geq 4$, $q \geq 3$. We used sizes and bounds from [2],[6, Table 2],[8]-[11],[13, Table 4.5],[15, Table I]. The result $19 \leq t_2(4,4)$ and a complete 21-cap in $PG(4,4)$ are obtained in [2].

Table 4: The sizes of the known complete k-caps in $PG(n,q)$, $n \geq 4$, $q \geq 3$

n	q	$t_2(n,q)$	Sizes k of the known complete caps with $t_2(n,q) \leq k \leq m'_2(n,q)$	$m'_2(n,q)$	$m_2(n,q)$	new
4	**3.**	11	$k=11$ and $16 \leq k \leq 19$	19	20	
4	4	$19 \leq$	$20 \leq k \leq 40$	40	41	[2]
4	5	$21 \leq$	$31 \leq k \leq 66$		≤ 88	⋆
4	7	$29 \leq$	$\mathbf{56} \leq k \leq 124$ and $k=126,132$		≤ 238	⋆
5	3	$20 \leq$	$k=22$ and $26 \leq k \leq 48$	48	56	⋆
5	4	$31 \leq$	$50 \leq k \leq 108$ and $k=112,126$		$\leq \mathbf{153}$	⋆
6	3	$34 \leq$	$k=44$ and $46 \leq k \leq 103$, $k=112$		≤ 136	⋆
6	4	$61 \leq$	$\mathbf{114} \leq k \leq 288$		$\leq \mathbf{607}$	⋆
7	3	$58 \leq$	$88 \leq k \leq 238$ and $243 \leq k \leq 248$		$\leq \mathbf{404}$	⋆
8	3	$100 \leq$	$176 \leq k \leq 532$ and $k=\mathbf{534}$		$\leq \mathbf{1208}$	⋆
9	3	$172 \leq$	$352 \leq k \leq 1214$ and $k=1216$		$\leq \mathbf{3247}$	⋆

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