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On the least covering radius of the binary linear codes of dimension 6

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Abstract. In this work a heuristic algorithm for obtaining lower bounds on the covering radius of a linear code is developed. Using this algorithm the least covering radii of the binary linear codes of dimension 6 are determined. Upper bounds for the least covering radii of binary linear codes of dimensions 8 and 9 are derived.

1 Introduction

In this work we address two problems: the mathematical question of determining $t_2[n,k]$, the smallest covering radius of any binary linear $[n,k]$ code, and the more practical problem of constructing codes having a specified length and dimension and the least covering radius. More precisely we determine all values of the function $t_2[n,6]$ and give constructions for such codes. An important part of the determination of the values of $t_2[n,6]$ is the suggested heuristic algorithm for computation of lower bound of the covering radius of a linear code. We also derive upper bounds for $t_2[n,8]$ and $t_2[n,9]$.

2 Some preliminary results

Let F_q^n be the n-dimensional vector space over the finite field with q elements. A *linear code* C is a k-dimensional subspace of F_q^n. The ball of radius t around a word $y \in F_q^n$ is defined by

$$\{x | x \in F_q^n, d(x,y) \le t\}.$$

Then the covering radius $R(C)$ of a code C is defined as the least possible integer number such that the balls of radius $R(C)$ around the codewords cover the whole F_q^n, i.e.

$$R(C) = \max_{x \in F_q^n} \min_{c \in C} d(x,c).$$

A coset of the code C defined by the vector $x \in F_q^n$ is the set $x + C = \{x + c \mid c \in C\}$. A coset leader of $x + C$ is a vector in $x + C$ of smallest weight. When the code is linear its covering radius is equal to the weight of the heaviest coset leader.

The function $t_q[n,k]$ is defined as the least value of $R(C)$ when C runs over the class of all linear $[n,k]$ codes over F_q for a given q.

Definition. [1] Let C be a binary code of length n and covering radius R. For $i=1,\ldots,n$ let $C_0^{(i)}$ (respectively $C_1^{(i)}$) denote the set of codewords in which the i-th coordinate is 0 (respectively 1). The integer

$$N^{(i)} = \max_{x\in F_2^n}\{d(x,C_0^{(i)}) + d(x,C_1^{(i)})\}$$

is called the *norm* of C with respect to the i-th coordinate and

$$N_{min} = \min_i N^{(i)}$$

is called the minimum norm of C. (We use the convention that $d(x,0)=\infty$.) The code C has norm N if $N_{min} \le N$ and the coordinates i for which $N^{(i)} < N$ are called *acceptable* with respect to N.

The code C is *normal* if it has norm $2R+1$. If $N^{(i)} \le 2R+1$, then we say that the coordinate i is acceptable with respect to $2R+1$, or that C is normal with respect to the i-th coordinate.

In the following theorem results about the normality of binary linear codes are summarized.

Theorem 1. [1] *If C is an $[n,k,d]$ code with $n\le 15$, $k\le 5$ or $n-k\le 9$, then C is normal.*

One of the main reasons for studying normal codes is the *amalgamated direct sum* (ADS) construction introduced by Graham and Sloane [2].

Theorem 2 [1] *Assume that A is a normal binary $[n_A,k_A]R_A$ code with the last coordinate acceptable, and B is a normal binary $[n_B,k_B]R_B$ code with the first coordinate acceptable. Then their amalgamated direct sum (ADS)*

$$A\dot{\oplus}B = \{(a,0,b)|(a,0)\in A,(0,b)\in B\}\cup\{(a,1,b)|(a,1)\in A,(1,b)\in B\}$$

is an $[n_A+n_B-1,k_A+k_B-1]R$ code with $R\le R_A+R_B$. More generally, if the norm of A with respect to the last coordinate is N_A and the norm of B with respect to the first coordinate is N_B, then the code $A\dot{\oplus}B$ has norm N_A+N_B-1 and hence covering radius at most $\frac{1}{2}(N_A+N_B-1)$. In particular, if the covering radius of $A\dot{\oplus}B$ equals R_A+R_B, then $A\dot{\oplus}B$ is normal and the overlapping coordinate is acceptable.

3 Least covering radius of the binary linear codes of dimension 6

The results about the least covering radius of binary linear codes are summarized in Table 7.1 from [1] where the exact values or bounds for $t_2[n,k]$ for codes of lengths up to 64 are given. We use this table as a source of our investigation and give the results for codes of dimension 6 in the following table.

Bounds on $t_2[n,6]$ for $n \leq 64$

n	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
$\mathbf{t_2[n,6]}$	1	1	1	2	2	3	3	3	4	4
n	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**
$\mathbf{t_2[n,6]}$	5	5	5	6	6	6-7	7	7-8	7-8	8-9
n	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**
$\mathbf{t_2[n,6]}$	8-9	9-10	9-10	9-11	10-11	10-12	11-12	11-13	11-13	12-14
n	**37**	**38**	**39**	**40**	**41**	**42**	**43**	**44**	**45**	**46**
$\mathbf{t_2[n,6]}$	12-14	13-15	13-15	14-16	14-16	14-17	15-17	15-18	16-18	16-19
n	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**
$\mathbf{t_2[n,6]}$	17-19	17-20	17-20	18-21	18-21	19-22	19-22	20-23	20-23	20-24
n	**57**	**58**	**59**	**60**	**61**	**62**	**63**	**64**		
$\mathbf{t_2[n,6]}$	21-24	21-25	22-25	22-26	23-26	23-27	23-27	24-28		

The values of $t_2[n,k]$ for codes of dimensions up to 5 are determined in [2] and also an upper bounds for $t_2[n,k]$ for codes of dimensions 6 and 7 are derived. Namely, in [2, Theorem 23] it is proved that

$$t_2[n,6] \leq \left\lfloor \frac{n-8}{2} \right\rfloor \text{ for } n \geq 18, \text{ and } t_2[n,7] \leq \left\lfloor \frac{n-9}{2} \right\rfloor \text{ for } n \geq 19.$$

In this work we show that the bound for codes of dimension 6 is sharp. The approach we use is similar to the approach from [2] and it is based on the determination of the covering radii of the projective codes of dimension 6. We will note that the covering radii of the binary projective codes of dimensions up to 5 are determined in [3] and [4].

Theorem 3. $t_2[n,6] = \left\lfloor \dfrac{n-8}{2} \right\rfloor$, *for* $n \geq 18$.

Proof. For codes of lengths 18-21 values of $t_2[n,6]$ are known and they fulfill the condition of the Theorem. For the rest of the codes of lengths up to 64 the upper bounds from the Table coincide with the value given in the theorem. What remains is to prove that these upper bounds are sharp. Let us consider the first open case $[22,6]$ codes. If a $[22,6]$ code C contains a repeated coordinate, then $R(C) \geq t_2[20,6]+1 = 7$. Thus, if there exits a $[22,6]$ code of covering radius 6 it must be a projective one. Classification of all binary projective codes of dimension up to 6 is done in [5]. We use the results of this classification where 2852541 nonequivalent binary $[22,6]$ codes are found, to show that there is no code of covering radius 6 among them. Therefore $t_2[22,6] = 7$. Let now C be a $[24,6]$ code. The same reasoning shows that

$t_2[24,6] = 8$ and as $t_2[25,6] \geq t_2[24,6]$ we get $t_2[25,6] = 8$. We repeat this until $t_2[54,6] = t_2[55,6] = 23$.

Let C be a $[56,6]$ code. If it contains a repeated coordinate, then $R(C) \geq t_2[54,6] + 1 = 23 + 1 = 24$. Otherwise, C is a shortened version of the $[63,6]$ Simplex code whit covering radius 31 and thus $R(C) \geq 31 - 7 = 24$. Therefore $t_2[56,6] = 24$ and $t_2[57,6] = 24$. Similarly $t_2[58,6] = t_2[59,6] = 25$, $t_2[60,6] = t_2[61,6] = 26$ and $t_2[62,6] = t_2[63,6] = 27$.

For $n \geq 64$, every $[n,6]$ code must contain a repeated coordinate and $t_2[n,6] \geq t_2[n-2,6]+1$, which implies $t_2[n,6] \geq \lfloor (n-8)/2 \rfloor$ for all n. Therefore the upper bound is sharp, which completes the proof. $\square$

The other aim of our investigation is to construct codes having covering radii equal to the least one. Here we will show how to do this. It is proved in [6] that the constructed in [2] $[14,6,5]3$ code is unique. A $[16,6]4$ or $[18,6]5$ code can be obtained by the $[14,6]3$ code by adding repeated coordinates. Again in [2] the generator matrix of $[19,6,7]5$ code is presented. As $[19,6]5$ codes must be projective ($t_2[19,5] \geq t_2[17,6] + 1 = 6$), we use the classification from [5] to determine the covering radii of all 366089 projective $[19,6]$ codes. It turned out that there is only one code with covering radius 5 and therefore the $[19,6,7]5$ code is unique. Then every $[n,6]$ code for $n > 19$ having the least covering radius can be obtained from the $[18,6]5$ or $[19,6]5$ codes by adding the necessary number of repeated coordinates. By adding repeated coordinates to the $[9,6]1$ code we can obtain $[11,6]2$, $[13,6]3$, $[15,6]4$ and $[17,6]5$ codes, and to the $[8,6]1$ code we can get $[10,6]2$ and $[12,6]3$ codes. We classify all $[8,6]$ and $[9,6]$ codes and among the 25 $[8,6]$ and 99 $[9,6]$ nonequivalent codes there are correspondingly 16 and 4 of covering radius 1.

4 A heuristic algorithm for lower bound of the covering radius of a linear code

In the proof of Theorem 1 we use a computer to show the nonexistence of codes of lengths $22 \leq n \leq 54$, n odd, and given covering radius. There are 236779414 such codes and if we try to determine their covering radii using one of the known for us algorithms it would take years. Here we present a heuristic algorithm which alows us to show the nonexistence of an $[n,k]R$ code C in a reasonable time.

The idea of the algorithm is as fast as possible to find a coset leader of the investigated code of weight greater than R, which means that the covering radius of the code is at least $R+1$. It starts with a randomly chosen vector c from a coset $K_c = \{c + C\}$. We use the evaluation function f to find the current best solution, where the aim is to minimize the number of vectors of minimum weight in the coset. The function $f = wt(K_c)2^k - A(K_c)$ depends on the weight of the coset $wt(K_c)$ and the number $A(K_c)$ of vectors of minimum eight in K_c. Then we search in the set of neighbors $N(c)$ consisting of vectors which differ from c in one coordinate. If in this procedure we obtain a coset of weight greater than R, we are done. Otherwise, we add some noise to c and again try to find a coset of weight greater than R.

Algorithm. LowerBoundCoveringRadius(R_{min})

```
c, c': vector;
br_0, br: integer;
{
  br_0 := 0;
   while br_0 < const_0
     {
         br_0 := br_0 + 1;
          br := 0;
          Select a feasible solution c;
          while br < const
         {
             br := br + 1;
             while exists c' ∈ N(c) such that f(c') > f(c) do  c := c';
              if wt(K_c) > R_min  break;
             Add some noise to c;
         }
     }
}
```

5 Upper bounds for the covering radii of linear codes of dimensions 8 and 9

Theorem 4. $t_2[n,8] \leq \left\lfloor \dfrac{n-10}{2} \right\rfloor$ *for* $n \geq 16$ *and* $t_2[n,9] \leq \lfloor \frac{n-12}{2} \rfloor$ *for* $n \geq 25$.

Proof. Let us consider the direct sum of two $[9,4]2$ normal codes. According to Theorem 9 from [2] we obtain $[18,8]4$ normal code and the existence of $[18+2i,8]4+i$ codes for $i>0$ follows from [2, Theorem 20]. The same way the direct sum of $[8,4]2$ and $[9,4]2$ normal codes gives $[17,8]4$ normal code and there exist $[17+2i,8]4+i$ codes for $i>0$. From [1, Table 7.1] we have $t_2[16,8]=3$ which completes the proof for the first upper bound.

Let us now consider the amalgamated direct sum of $[7,4]1$ and $[14,6]3$ normal codes. The result is a $[20,9]4$ code which according to Theorem 2 is normal. A $[25,9]6$ normal code can be obtained by an ADS of $[7,4]1$ code and the constructed in [2] and proved to be unique in this work $[19,6]5$ code. As in the previous case, we can conclude that the $[20+2i, 9]4+i$ and $[25+2i, 9]6+i$ codes for $i > 0$ exist and the upper bound for $t_2[n, 9]$ follows. □

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