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Nonexistence results for spherical 7-designs

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Spherical designs

Definition 1. *A spherical τ-design $C \subset \mathbb{S}^{n-1}$ is a finite nonempty subset of $\mathbb{S}^{n-1}$ such that for every point $x \in \mathbb{S}^{n-1}$ and for every real polynomial $f(t)$ of degree at most τ, the equality*

$$\sum_{y \in C} f(\langle x, y \rangle) = f_0 |C|. \qquad (1)$$

holds, where f_0 is the first coefficient in the expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ in terms of the Gegenbauer polynomials. The number τ is called strength of C.

Spherical designs

When $x \in C$*, (1) becomes*

$$\sum_{y \in C \setminus \{x\}} f(\langle t_i(x) \rangle) = f_0|C| - f(1), \qquad (2)$$

where $t_1(x) \leq t_2(x) \leq \cdots \leq t_{|C|-1}(x)$ *are the inner products of* $x \in C$ *with all other points of* C*.*

Problems

- **Problem A.** Obtain bounds on inner products of spherical τ-design $C \subset \mathbb{S}^{n-1}$ for fixed strength τ, dimension n and cardinality $M = |C|$.
- **Problem B.** Decide whether a τ-design on $\mathbb{S}^{n-1}$ of cardinality $M = |C|$ exists for fixed strength τ, dimension n and cardinality M.

Some known results

Denote $B(n,\tau) = \min\{|C| : C \in \mathbb{S}^{n-1} \text{ is a } \tau\text{-design}\}$.

Delsarte-Goethals-Seidel bound, (Geom. Dedicata, 1977)

$$B(n,\tau) \geq D(n,\tau) = \begin{cases} 2\binom{n+k-2}{n-1}, & \textit{if} \quad \tau = 2k-1, \\ \binom{n+k-1}{n-1} + \binom{n+k-2}{n-1}, & \textit{if} \quad \tau = 2k. \end{cases}$$

Some known results

- **Reznick, (Lin. Alg. Appl., 1995)**
  Constructions of spherical 5-designs in three dimensions for cardinalities 12, 16, 18, and ≥ 20.

- **Bajnok, (Graphs Combin.,1998),(Des. Codes Crypt., 2000)** Constructions of 3-designs on $\mathbb{S}^{n-1}$ with all admissible even cardinalities (i.e. $\geq 2n$) and all odd cardinalities greater than or equal to $5n/2$ for $n \geq 6$, and to 11 for $n = 3, 4$, and 15 for $n = 5$.

Some known results

- **Fazekas-Levenshtein, (J. Combin. Theory, 1997)**

  Restrictions on the structure of spherical designs.

- **Boyvalenkov-Danev-Nikova, (Discr. Comput. Geom., 1998)**

  Nonexistence results of spherical designs with odd strength and odd $|C|$.

  Complete solution of Problem B for $\tau = 3$ in dimensions $n = 4$ and 6.

Some known results

- **Boumova-Boyvalenkov-Danev, (Europ. J. Combin., 1999)**

  Necessary Condition: *If* $C \subset \mathbb{S}^{n-1}$ *is a* τ*-design with odd* $\tau = 2e - 1$ *and odd* $|C|$ *then* $\rho_0|C| \geq 2$*.*

- **Boumova-Boyvalenkov-Danev, (Proc. CTF, 2002)**

  50 new nonexistence results of spherical 3-designs with odd $|C|$, such that $\rho_0|C| \geq 2$ and $3 \leq n \leq 50$.

  Complete solution of Problem B for $\tau = 3$ in dimensions $n = 9$ and 10.

Some known results

- **Boumova-Boyvalenkov-Kulina-Stoyanova, (Proc. OCRT, 2007)**

  42 (out of all possible 42) new nonexistence results of spherical 5-designs with odd $|C|$, such that $2 \leq \rho_0|C| < 3$ and $5 \leq n \leq 25$.

- $\Rightarrow \quad \rho_0|C| \geq 2 \quad$ can be replaced by $\quad \rho_0|C| \geq 3$ (for 5-designs in dimensions $5 \leq n \leq 25$.)

Some known results

- **Boumova-Boyvalenkov-Kulina-Stoyanova, (submitted), 2008**

  35 (out of all possible 47) new nonexistence results of spherical 3-designs with odd $|C|$, such that $2 \leq \rho_0|C| < 3$, $2\alpha_0^2 - 1 > \alpha_1$ and $3 \leq n \leq 50$.

  Complete solution of Problem B for $\tau = 3$ in dimensions $n = 8, 13, 14$ and 18.

  A table for Problem B for $\tau = 3$ can be found:

  $http://www.fmi.uni-sofia.bg/algebra/publications/stoyanova/table.html$

Problem B for $\tau = 7$

- **In this talk:** $(290 + 18)$ new nonexistence results for 7-designs in dimensions $n \leq 20$.

- Let $C \subset \mathbb{S}^{n-1}$ be a 7-design. Then

$$|C| \geq B(n,7) = 2\binom{n+2}{3} + 1 = \frac{n(n+1)(n+2)}{3} + 1$$

by the Delsarte-Goethals-Seidel bound.

Problem B for $\tau = 7$

$\{\alpha_i\}$, $i = 0, 1, 2, 3$, $(s = \alpha_3)$ roots of

$$\begin{aligned}(n+4)(2+n)(n^3+6n^2+5n-6|C|)x^4 - \\ n(n^2-1)(n+2)(n+2)x^3 - \\ 9(n+2)(n^3+4n^2+3n-4|C|)x^2 + \\ 3n(n^2-1)(n+2)x+6n^3+18n^2+12n-18|C| = 0.\end{aligned}$$

$$\rho_0|C| = -\frac{(1-\alpha_1^2)(1-\alpha_2^2)(1-\alpha_3^2)}{\alpha_0(\alpha_0^2-\alpha_1^2)(\alpha_0^2-\alpha_2^2)(\alpha_0^2-\alpha_3^2)}.$$

Problem B for $\tau = 7$

- **Lemma 1.** (BBD) *Let $C \subset \mathbb{S}^{n-1}$ be a τ-design with odd $\tau = 2e - 1$. Then for every point $x \in C$ we have $t_1(x) \leq \alpha_0$ and $t_{|C|-1}(x) \geq \alpha_{e-1}$. In particular, we have $s(C) \geq \alpha_{e-1}$. If $|C|$ is odd then there exist a point $x \in C$ such that $t_2(x) \leq \alpha_0$.*
- **Lemma 2.** (BBD) *Let $C \subset \mathbb{S}^{n-1}$ be a τ-design with odd $\tau = 2e - 1$ and odd cardinality $|C|$. Then there exist three distinct points $x, y, z \in C$ such that $t_1(x) = t_1(y)$ and $t_2(x) = t_1(z)$. Moreover, we have $t_{|C|-1}(z) \geq \max\{\alpha_{e-1}, 2\alpha_0^2 - 1\}$. In particular, we have $s(C) \geq \max\{\alpha_{e-1}, 2\alpha_0^2 - 1\}$.*

Problem B for $\tau = 7$

- **Theorem 3.** (BBD) *If $C \subset \mathbb{S}^{n-1}$ is a τ-design with odd $\tau = 2e - 1$ and odd $|C|$ then $\rho_0|C| \geq 2$.*
- **Notation:** $U_{\tau,i}(x)$ (respectively $L_{\tau,i}(x)$) for any upper (resp. lower) bound on the inner product $t_i(x)$. When a bound does not depend on x we omit x in the notation.
- For example, the first bound from Lemma 1 is $t_1(x) \leq U_{\tau,1} = \alpha_0$ and the last bound from Lemma 2 is $t_{|C|-1}(z) \geq L_{\tau,|C|-1}(z) = \max\{\alpha_{e-1}, 2\alpha_0^2 - 1\}$.

Sketch of the algorithm

- Let $C \subset \mathbb{S}^{n-1}$ be a 7-design with odd $|C| \geq 2\binom{n+2}{3} + 1$ and $2\alpha_0^2 - 1 > \alpha_3$.
- A special triple of points $x, y, z \in C$ exists. (Lemma 2)
- We focus on the inner products in $I(x)$ and $I(z)$, plus a point $u \in C$ such that $\langle u, z \rangle = t_2(z)$.

Sketch of the algorithm

- Bounds:

$$L_{7,1}(z) \le t_1(z),$$

$$t_2(z) \le U_{7,2}(z),$$

$$L_{7,3}(z) \le t_3(z).$$

- Sometimes: $t_1(z) \ge L_{7,1}(z) > U_{7,1}(z) = \alpha_0 \ge t_1(z)$
  or $t_2(z) \le U_{7,2}(z) < L_{7,1}(z) \le t_1(z)$.
  (in cases where $\rho_0|C|$ is close from above to 2).
- When $U_{7,2}(z) \ge U_{7,1}(z) \ge L_{7,1}(z)$, we consider two cases for the location of $t_2(z)$ with respect to α_0.

Sketch of the algorithm

- **Case 1.** If $t_2(z) \in [\alpha_0, U_{7,2}(z)]$.

  We obtain new upper bound $t_1(z) \leq U_{7,1}(z) < \alpha_0$ which can be used for obtaining a contradiction.

  If necessary (in a few cases) we organize an iteration procedure.

Sketch of the algorithm

- **Case 2.** If $t_2(z) \in [t_1(z), \alpha_0]$.
  For the point $u \in C$: $t_2(z) = \langle z, u \rangle \leq \alpha_0$.

  Theorem 3. (BBKS) *There exists a special quadruple* $\{x, y, z, u\} \subset C$ *such that* $\max\{t_{|C|-2}(z), t_{|C|-2}(x)\} \geq 2\alpha_0^2 - 1$.
  In both cases we continue with new bounds $L_{7,3}(z) \leq t_3(z)$ and $t_1(z) \leq U_{7,1}(z)$ which can be used for obtaining a contradiction.

  In some cases we need more careful consideration of the location of some inner products and iteration procedures.

The new nonexistence results

- There are 291 open cases in dimensions $3 \le n \le 20$ with odd $|C|$ such that $2 \le \rho_0|C| < 3$.
- In every such case we have $2\alpha_0^2 - 1 > \alpha_3$, i.e. $t_{|C|-1}(z) \ge L_{3,|C|-1}(z) = 2\alpha_0^2 - 1$ by Lemma 2.
- Applying our algorithm we obtain nonexistence in all cases with only one exception – the case $n = 4$, $|C| = 43$.
- There are 18 cases of nonexistence with $\rho_0|C| \ge 3$ as well.
- $\Rightarrow \quad \rho_0|C| \ge 3$ in dimensions $5 \le n \le 20$ and $\tau = 7$.

Lower bounds on $B_{odd}(n, 7)$

n	BDN	BBD	This paper $\rho_0\|C\| \in [2, 3)$	This paper $\rho_0\|C\| \geq 3$
3		23	23	
4		43	43	
5	73	75	77	
6	117	119	123	
7	173	177	183	
8		253	261	
9		347	359	
10		463	477	
11		601	619	621
12		765	789	
13		957	985	987
14		1175	1213	1215
15		1427	1471	1475
16		1713	1767	1769
17		2031	2097	2101
18		2393	2467	2473
19		2791	2879	2885
20		3233	3333	3341

THANK YOU FOR YOUR ATTENTION !