Nonexistence results for spherical 7-designs

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Definition 1. A spherical $\tau$-design $C \subset \mathbb{S}^{n-1}$ is a finite nonempty subset of $\mathbb{S}^{n-1}$ such that for every point $x \in \mathbb{S}^{n-1}$ and for every real polynomial $f(t)$ of degree at most $\tau$, the equality

$$\sum_{y \in C} f(\langle x, y \rangle) = f_0 |C|.$$  \hspace{1cm} (1)

holds, where $f_0$ is the first coefficient in the expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ in terms of the Gegenbauer polynomials. The number $\tau$ is called strength of $C$. 

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When $x \in C$, (1) becomes

$$\sum_{y \in C \backslash \{x\}} f(\langle t_i(x) \rangle) = f_0 |C| - f(1),$$

(2)

where $t_1(x) \leq t_2(x) \leq \cdots \leq t_{|C|-1}(x)$ are the inner products of $x \in C$ with all other points of $C$. 

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Problems

- **Problem A.** Obtain bounds on inner products of spherical $\tau$-design $C \subset \mathbb{S}^{n-1}$ for fixed strength $\tau$, dimension $n$ and cardinality $M = |C|$.

- **Problem B.** Decide whether a $\tau$-design on $\mathbb{S}^{n-1}$ of cardinality $M = |C|$ exists for fixed strength $\tau$, dimension $n$ and cardinality $M$. 

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Some known results

Denote $B(n, \tau) = \min\{|C| : C \in S^{n-1} \text{ is a } \tau\text{-design}\}$.


$$B(n, \tau) \geq D(n, \tau) = \begin{cases} 2\binom{n+k-2}{n-1}, & \text{if } \tau = 2k - 1, \\ \binom{n+k-1}{n-1} + \binom{n+k-2}{n-1}, & \text{if } \tau = 2k. \end{cases}$$
Some known results

  Constructions of spherical 5-designs in three dimensions for cardinalities 12, 16, 18, and $\geq 20$.

  Constructions of 3-designs on $S^{n-1}$ with all admissible even cardinalities (i.e. $\geq 2n$) and all odd cardinalities greater than or equal to $5n/2$ for $n \geq 6$, and to 11 for $n = 3, 4$, and 15 for $n = 5$. 
Some known results

- **Fazekas-Levenshtein, (J. Combin. Theory, 1997)**
  Restrictions on the structure of spherical designs.

  Nonexistence results of spherical designs with odd strength and odd $|C|$.
  Complete solution of Problem B for $\tau = 3$ in dimensions $n = 4$ and 6.
Some known results

  
  Necessary Condition: *If* $C \subset \mathbb{S}^{n-1}$ *is a* $\tau$-*design with odd* $\tau = 2e - 1$ *and odd* $|C|$ *then* $\rho_0 |C| \geq 2$.

- **Boumova-Boyvalenkov-Danev**, *(Proc. CTF, 2002)*
  
  50 new nonexistence results of spherical 3-designs with odd $|C|$, such that $\rho_0 |C| \geq 2$ and $3 \leq n \leq 50$.
  
  Complete solution of Problem B for $\tau = 3$ in dimensions $n = 9$ and 10.
Some known results

- Boumova-Boyvalenkov-Kulina-Stoyanova, (Proc. OCRT, 2007)

42 (out of all possible 42) new nonexistence results of spherical 5-designs with odd $|C|$, such that $2 \leq \rho_0 |C| < 3$ and $5 \leq n \leq 25$.

- $\Rightarrow \rho_0 |C| \geq 2$ can be replaced by $\rho_0 |C| \geq 3$
  (for 5-designs in dimensions $5 \leq n \leq 25$.)
Some known results

- **Boumova-Boyvalenkov-Kulina-Stoyanova**, (submitted), 2008

35 (out of all possible 47) new nonexistence results of spherical 3-designs with odd $|C|$, such that $2 \leq \rho_0 |C| < 3$, $2\alpha_0^2 - 1 > \alpha_1$ and $3 \leq n \leq 50$.

Complete solution of Problem B for $\tau = 3$ in dimensions $n = 8, 13, 14$ and 18.

A table for Problem B for $\tau = 3$ can be found:

http://www.fmi.uni-sofia.bg/algebra/publications/stoyanova/table.html
In this talk: (290 + 18) new nonexistence results for 7-designs in dimensions $n \leq 20$.

Let $C \subset S^{n-1}$ be a 7-design. Then

$$|C| \geq B(n, 7) = 2 \binom{n+2}{3} + 1 = \frac{n(n+1)(n+2)}{3} + 1$$

by the Delsarte-Goethals-Seidel bound.
Problem B for $\tau = 7$

\[ \{\alpha_i\}, \ i = 0, 1, 2, 3, \ (s = \alpha_3) \quad \text{roots of} \]

\[ (n + 4)(2 + n)(n^3 + 6n^2 + 5n - 6|C|)x^4 - \]
\[ n(n^2 - 1)(n + 2)(n + 2)x^3 - \]
\[ 9(n + 2)(n^3 + 4n^2 + 3n - 4|C|)x^2 + \]
\[ 3n(n^2 - 1)(n + 2)x + 6n^3 + 18n^2 + 12n - 18|C| = 0. \]

\[ \rho_0|C| = -\frac{(1 - \alpha_1^2)(1 - \alpha_2^2)(1 - \alpha_3^2)}{\alpha_0(\alpha_0^2 - \alpha_1^2)(\alpha_0^2 - \alpha_2^2)(\alpha_0^2 - \alpha_3^2)}. \]
Lemma 1. (BBD) Let $C \subset \mathbb{S}^{n-1}$ be a $\tau$-design with odd $\tau = 2e - 1$. Then for every point $x \in C$ we have $t_1(x) \leq \alpha_0$ and $t_{|C|-1}(x) \geq \alpha_{e-1}$. In particular, we have $s(C) \geq \alpha_{e-1}$. If $|C|$ is odd then there exist a point $x \in C$ such that $t_2(x) \leq \alpha_0$.

Lemma 2. (BBD) Let $C \subset \mathbb{S}^{n-1}$ be a $\tau$-design with odd $\tau = 2e - 1$ and odd cardinality $|C|$. Then there exist three distinct points $x, y, z \in C$ such that $t_1(x) = t_1(y)$ and $t_2(x) = t_1(z)$. Moreover, we have $t_{|C|-1}(z) \geq \max\{\alpha_{e-1}, 2\alpha_0^2 - 1\}$. In particular, we have $s(C) \geq \max\{\alpha_{e-1}, 2\alpha_0^2 - 1\}$.
Theorem 3. (BBD) If $C \subset S^{n-1}$ is a $\tau$-design with odd $\tau = 2e - 1$ and odd $|C|$ then $\rho_0 |C| \geq 2$.

Notation: $U_{\tau,i}(x)$ (respectively $L_{\tau,i}(x)$) for any upper (resp. lower) bound on the inner product $t_i(x)$. When a bound does not depend on $x$ we omit $x$ in the notation.

For example, the first bound from Lemma 1 is $t_1(x) \leq U_{\tau,1} = \alpha_0$ and the last bound from Lemma 2 is $t_{|C|-1}(z) \geq L_{\tau,|C|-1}(z) = \max\{\alpha_{e-1}, 2\alpha_0^2 - 1\}$. 
Let $C \subset \mathbb{S}^{n-1}$ be a 7-design with odd $|C| \geq 2\binom{n+2}{3} + 1$ and $2\alpha_0^2 - 1 > \alpha_3$.

A special triple of points $x, y, z \in C$ exists. (Lemma 2)

We focus on the inner products in $l(x)$ and $l(z)$, plus a point $u \in C$ such that $\langle u, z \rangle = t_2(z)$. 

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Sketch of the algorithm

- **Bounds:**
  \[
  L_{7,1}(z) \leq t_1(z), \\
  t_2(z) \leq U_{7,2}(z), \\
  L_{7,3}(z) \leq t_3(z).
  \]

- **Sometimes:**
  \[
  t_1(z) \geq L_{7,1}(z) > U_{7,1}(z) = \alpha_0 \geq t_1(z) \\
  \text{or} \quad t_2(z) \leq U_{7,2}(z) < L_{7,1}(z) \leq t_1(z).
  \]
  (in cases where \( \rho_0 |C| \) is close from above to 2).

- **When** \( U_{7,2}(z) \geq U_{7,1}(z) \geq L_{7,1}(z) \), we consider two cases for the location of \( t_2(z) \) with respect to \( \alpha_0 \).
Case 1. If $t_2(z) \in [\alpha_0, U_{7,2}(z)]$.

We obtain new upper bound $t_1(z) \leq U_{7,1}(z) < \alpha_0$ which can be used for obtaining a contradiction.

If necessary (in a few cases) we organize an iteration procedure.
Case 2. If $t_2(z) \in [t_1(z), \alpha_0]$. 
For the point $u \in C$: $t_2(z) = \langle z, u \rangle \leq \alpha_0$.

Theorem 3. (BBKS) There exists a special quadruple 
\{x, y, z, u\} \subset C such that 
\[ \max \{t_{|C|/2}(z), t_{|C|/2}(x)\} \geq 2\alpha_0^2 - 1. \]

In both cases we continue with new bounds $L_{7,3}(z) \leq t_3(z)$ and $t_1(z) \leq U_{7,1}(z)$ which can be used for obtaining a contradiction.

In some cases we need more careful consideration of the location of some inner products and iteration procedures.
The new nonexistence results

- There are 291 open cases in dimensions $3 \leq n \leq 20$ with odd $|C|$ such that $2 \leq \rho_0 |C| < 3$.

- In every such case we have $2\alpha_0^2 - 1 > \alpha_3$, i.e. $t_{|C|-1}(z) \geq L_{3,|C|-1}(z) = 2\alpha_0^2 - 1$ by Lemma 2.

- Applying our algorithm we obtain nonexistence in all cases with only one exception – the case $n = 4, |C| = 43$.

- There are 18 cases of nonexistence with $\rho_0 |C| \geq 3$ as well.

- $\Rightarrow \rho_0 |C| \geq 3$ in dimensions $5 \leq n \leq 20$ and $\tau = 7$. 
Lower bounds on $B_{odd}(n, 7)$

| $n$ | BDN | BBD | This paper $\rho_0|C| \in [2, 3)$ | This paper $\rho_0|C| \geq 3$ |
|-----|-----|-----|-------------------------------|-------------------------------|
| 3   | 23  | 23  |                               |                               |
| 4   | 43  | 43  |                               |                               |
| 5   | 73  | 75  | 77                           |                               |
| 6   | 117 | 119 | 123                          |                               |
| 7   | 173 | 177 | 183                          |                               |
| 8   | 253 | 261 |                               |                               |
| 9   | 347 | 359 |                               |                               |
| 10  | 463 | 477 |                               |                               |
| 11  | 601 | 619 | 621                          |                               |
| 12  | 765 | 789 |                               |                               |
| 13  | 957 | 985 | 987                          |                               |
| 14  | 1175| 1213| 1215                         |                               |
| 15  | 1427| 1471| 1475                         |                               |
| 16  | 1713| 1767| 1769                         |                               |
| 17  | 2031| 2097| 2101                         |                               |
| 18  | 2393| 2467| 2473                         |                               |
| 19  | 2791| 2879| 2885                         |                               |
| 20  | 3233| 3333| 3341                         |                               |
THANK YOU FOR YOUR ATTENTION!