On the Erasure-Correcting Capabilities of Low-Complexity Decoded LDPC Codes with Constituent Hamming Codes

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ACCT '08, Pamporovo, Bulgaria, June 2008

Outline

- ▷ Hamming code-based low-density parity-check (H-LDPC) codes
- ▷ Erasure correcting capabilities of Hamming codes
- ▷ Iterative erasure-correction algorithm for H-LDPC codes
- ▷ Asymptotic performance
- ▷ Numerical results

Hamming Code-Based LDPC (H-LDPC) Codes

- Parity-check matrix of Gallager's LDPC codes:

$$\boldsymbol{H} = \begin{pmatrix} \pi_1(\boldsymbol{H}_{\mathrm{b}}) \\ \pi_2(\boldsymbol{H}_{\mathrm{b}}) \\ \vdots \\ \pi_\ell(\boldsymbol{H}_{\mathrm{b}}) \end{pmatrix}_{\ell b \times b n_0}$$

where

$$\boldsymbol{H}_{\mathrm{b}} = \begin{pmatrix} 1\,1\,\ldots\,1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & 1\,1\,\ldots\,1 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & 1\,1\,\ldots\,1 \end{pmatrix}_{b \times b n_0}$$

- $(n_0, n_0 - 1)$ single parity-check (SPC) codes are constituent codes
- ℓ random column permutations of $\boldsymbol{H}_{\mathrm{b}}$ form layers of $\boldsymbol{H}$
- Code rate is $R \geq 1 - \frac{\ell b}{b n_0}$

Hamming Code-Based LDPC (H-LDPC) Codes

- Parity-check matrix of H-LDPC codes:

$$\boldsymbol{H} = \begin{pmatrix} \pi_1(\boldsymbol{H}_{\mathrm{b}}) \\ \pi_2(\boldsymbol{H}_{\mathrm{b}}) \\ \vdots \\ \pi_\ell(\boldsymbol{H}_{\mathrm{b}}) \end{pmatrix}_{\ell b(n_0-k_0)\times bn_0}$$

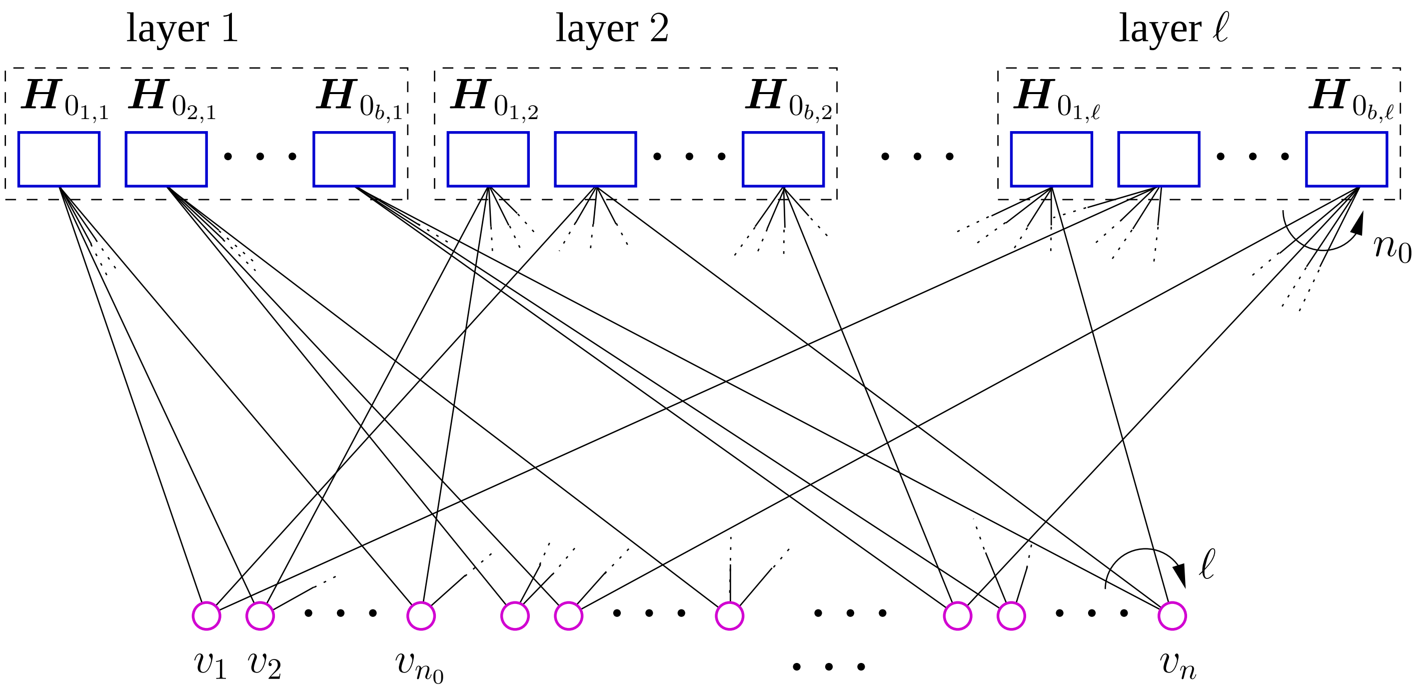
where

$$\boldsymbol{H}_{\mathrm{b}} = \begin{pmatrix} \boldsymbol{H}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{H}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{H}_0 \end{pmatrix}_{b(n_0-k_0)\times bn_0}$$

- (n_0, k_0) Hamming codes are constituent codes
- ℓ random column permutations of $\boldsymbol{H}_{\mathrm{b}}$ form layers of $\boldsymbol{H}$
- Code rate is $R \geq 1 - \frac{\ell b(n_0-k_0)}{bn_0} \Rightarrow \frac{k_0}{n_0} > 1 - \frac{1}{\ell}$

Hamming Code-Based LDPC (H-LDPC) Codes

- Bipartite Tanner graph of H-LDPC codes:



- Constraint nodes have degree n_0 and represent constituent Hamming codes.
  Constituent parity-check matrices $\boldsymbol{H}_{0_{j,k}}$ are all equal up to column permutations.
- Variable nodes have degree ℓ and represent codesymbols.
  Each variable node is connected to exactly one constraint node in each layer.

Erasure-Correcting Capabilities of Hamming Codes

- Parameters of the (n_0, k_0) Hamming code:
  - $n_0 = 2^m - 1$, $k_0 = n_0 - m$, $m \geq 2$, $d_0 = 3$
  - columns of $\boldsymbol{H}_0$ are all nonzero binary m-tuples
- When communicating over the binary erasure channel (BEC):
  - all erasure patterns with $d_0 - 1 = 2$ or fewer erasures are correctable
  - some erasure patterns with up to m erasures are correctable
- An erasure pattern with $\tau \leq m$ erasures is correctable if the $m \times \tau$ matrix $\boldsymbol{M}$, whose columns are the τ columns of $\boldsymbol{H}_0$ corresponding to the erased positions, has $\operatorname{rank}(\boldsymbol{M}) = \tau$.

  Lemma: The number of correctable erasure patterns with $\tau \leq m$ erasures is

$$\mathcal{M}(\tau, m) = \frac{\prod_{i=0}^{\tau-1} (2^m - 2^i)}{\tau!}$$

  Corollary: The generating function for the number of correctable erasure patterns is

$$g_1(s, n_0) = \sum_{\tau=1}^{m} \mathcal{M}(\tau, m) s^\tau = \binom{n_0}{1} s + \binom{n_0}{2} s^2 + \sum_{\tau=3}^{m} \frac{\prod_{i=0}^{\tau-1} (2^m - 2^i)}{\tau!} s^\tau$$

Iterative Erasure-Correcting Algorithm for H-LDPC Codes

- Consider communication over a BEC using an H-LDPC code of length $n = bn_0$ with constituent Hamming codes of length $n_0 = 2^m - 1$. Let $\boldsymbol{r}$ denote the received sequence.
- Iterative decoding algorithm $\mathscr{A}$: For every iteration $i = 1, 2, ..., i_{\max}$:

  (1) For the tentative sequence $\boldsymbol{r}^{(i)}$ (where $\boldsymbol{r}^{(1)} = \boldsymbol{r}$), select constituent codes $\mathcal{C}_{0j,l}$ with $\tau_{j,l}$ erasures, $j = 1, 2, ..., b$, $l = 1, 2, ..., \ell$, such that:

  Variant $\mathscr{A}_1$: $\tau_{j,l} < d_0 = 3$

  Variant $\mathscr{A}_2$: $\tau_{j,l} \leq m$

  (2) Assuming 0s on the erased positions, compute the syndromes $\boldsymbol{s}_{j,l}$ for the selected constituent codes.

  (3) For each selected constituent code $\mathcal{C}_{0j,l}$, construct the $m \times \tau_{j,l}$ matrix $\boldsymbol{M}_{j,l}$. If $\operatorname{rank}(\boldsymbol{M}_{j,l}) = \tau_{j,l}$, the code is affected by a correctable erasure pattern.

  (4) For every constituent code affected by a correctable erasure pattern find the erased tuple $\boldsymbol{x}_{j,l}$ by solving

$$\boldsymbol{x}_{j,l}\boldsymbol{M}_{j,l}^{\mathrm{T}} = \boldsymbol{s}_{j,l}$$

  Replace the erasures in $\boldsymbol{r}^{(i)}$ with the so-found code symbols. This yields $\boldsymbol{r}^{(i+1)}$.

Iterative Erasure-Correcting Algorithm for H-LDPC Codes

- Decoding complexity:

  Lemma:

  If an erasure pattern is such that in each iteration of the algorithm $\mathscr{A}$, the fraction of constituent codes affected by correctable erasures is

  $$\alpha > 0$$

  then the algorithm $\mathscr{A}$ recovers the transmitted codeword after $\mathcal{O}(\log n)$ iterations.

  Thus, the overall decoding complexity is $\mathcal{O}(n \log n)$.

Asymptotic Performance

- *Theorem:*

In the ensemble $\mathscr{C}(n_0, \ell, b)$ of H-LDPC codes, there exist codes (with probability p, where $\lim_{n\to\infty} p = 1$), which can, with decoding complexity $\mathcal{O}(n \log n)$, correct any erasure pattern with up to $W = \omega_\alpha n$ erasures. The value ω_α is the largest root of the equation

$$h(\omega) - \ell F(\alpha, \omega, n_0) = 0$$

where $h(\omega)$ is the binary entropy function, and the function $F(\alpha, \omega, n_0)$ is given by

$$F(\alpha, \omega, n_0) \triangleq h(\omega) - \frac{h(\alpha\omega n_0)}{n_0} + \max\left\{\omega \log_2 s - \frac{1}{n_0}\log_2(g_0(s, n_0)) - \alpha\omega \log_2\left(\frac{g_1(s, n_0)}{g_0(s, n_0)}\right)\right\}$$

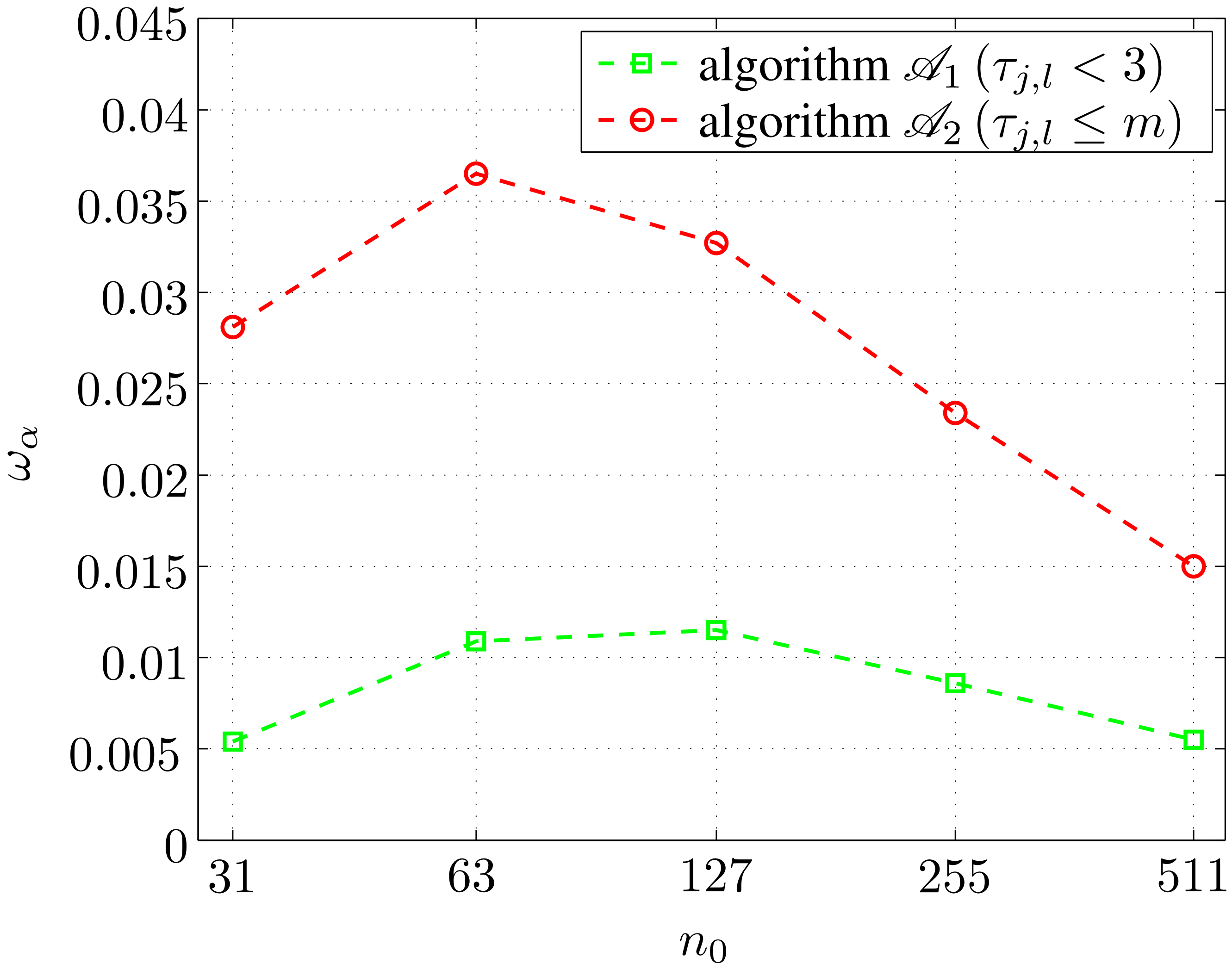
where $\alpha > 0$ and the maximization is performed over all s such that

$$\frac{\alpha\omega n_0}{1 - \alpha\omega n_0} \leq \frac{g_1(s, n_0)}{g_0(s, n_0)}.$$

The function $g_1(s, n_0)$ is the generating function of all the erasure patterns that are correctable by the constituent Hamming codes, and $g_0(s, n_0)$ is the generating function of the uncorrectable erasure patterns, $g_0(s, n_0) = (1 + s)^{n_0} - g_1(s, n_0)$.

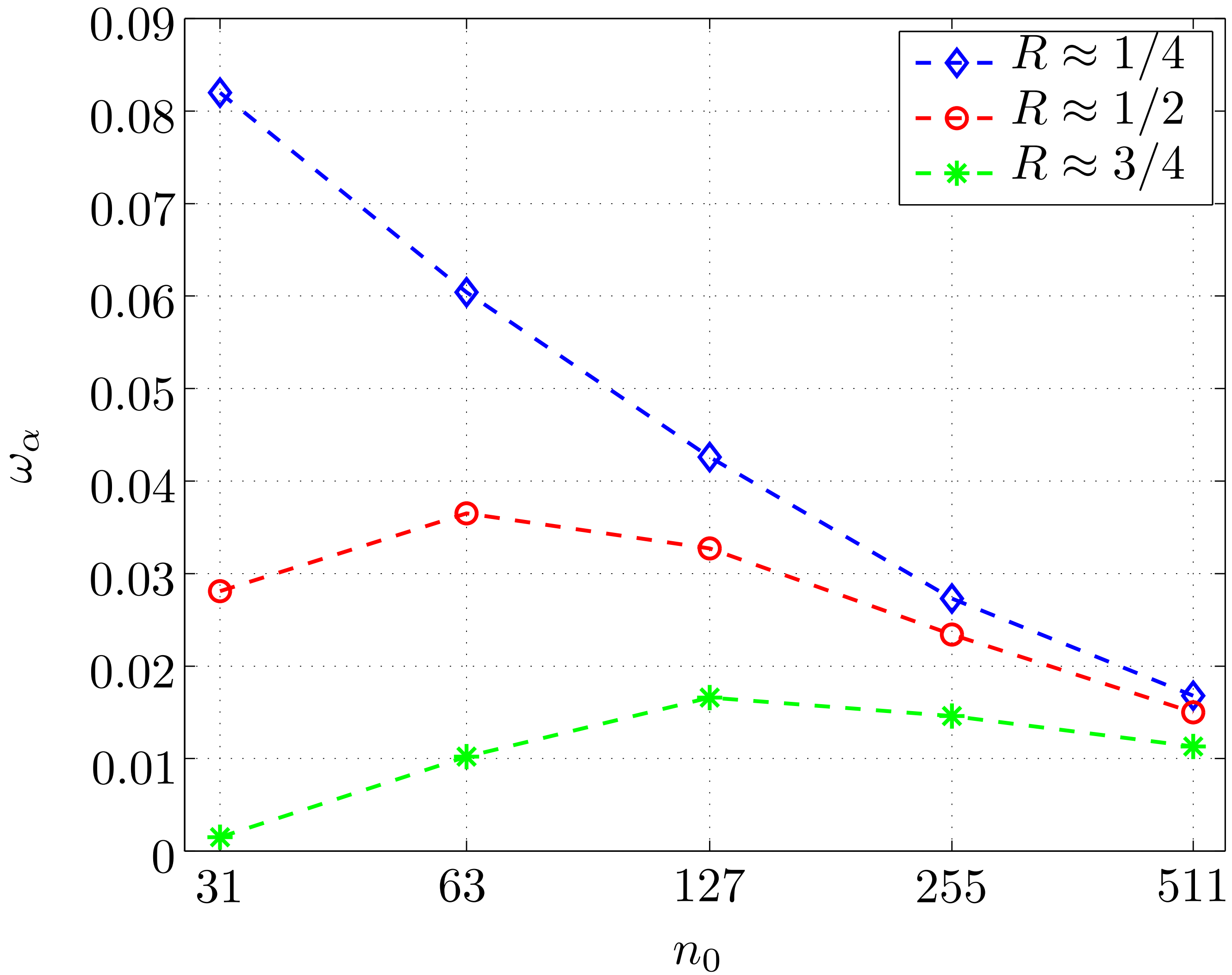
Numerical Results

- Code ensembles of rates $R \approx \frac{1}{2}$, $\alpha = 10^{-4}$, decoded with $\mathscr{A}_1$ and $\mathscr{A}_2$



Numerical Results

- Code ensembles of rates $R \approx \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, $\alpha = 10^{-4}$, decoded with $\mathscr{A}_2$



Conclusions

- ▷ Random Hamming code-based LDPC codes were used over the binary erasure channel.
- ▷ Constituent Hamming codes of length $n_0 = 2^m - 1$ can correct up to m erasures (if the corresponding columns of the parity-check matrix are linearly independent).
- ▷ Simple iterative decoding algorithm of complexity $\mathcal{O}(n \log n)$ was considered.
- ▷ Existence of H-LDPC codes capable of correcting $\mathcal{O}(n)$ erasures with complexity $\mathcal{O}(n \log n)$ was proved and verified numerically for several code ensembles.