On the switching construction of Steiner quadruple systems

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Summary The structure of Steiner quadruple system $S(v, 4, 3)$ of full 2-rank $v - 1$ is considered. It is shown that there are two types (induced and singular) of such systems. It is shown that induced Steiner systems can be obtained from Steiner systems $S(v, 4, 3)$ of 2-rank $v - 2$ by switching construction which is introduced here.

Introduction. A Stener system $S(n,k,t)$ is a pair (J,B) where J is a v-set and B is a collection of k-subsets of J such that every t-subset of J is contained in exactly one member of B. The necessary condition for existence of an SQS(v) is that $v \equiv 2$ or $4 \pmod 6$. Hanani [1960] proved that the necessary condition for the existence of an $S(v,4,3)$ is also sufficient.

A Steiner system $S(v,4,3)$ is called resolvable if it can be split into mutually non-overlapping sets so that every set is a Steiner system $S(v,4,1)$.

We consider the Steiner systems $S(v,4,3)$ of full 2-rank, i.e. of rank $v-1$ over F_2. Any such system is one of two types, which we call *induced* and *singular*. The induced systems can be obtained by a switching operation (which we derive here) from Steiner systems $S(v,4,3)$ of 2-rank $v-2$. This operation allows to construct Steiner systems of rank $r+1$ from systems of rank r. This operation is also interesting for the construction of resolvable Steiner systems. Namely, it keeps this property under certain conditions.

The case $n = 16$ is considered in details. In particular, we found exactly 305616 non-isomorphic induced Steiner systems $S(16,4,3)$, which were constructed by the switching operation from all 708103 non-isomorphic systems $S(16,4,3)$ of rank 14.

2. Preliminary results.

Let $E = \{0,1\}$. Denote a binary code C with length n, with minimum distance d and cardinality N as a (n,d,N)-code. Denote by $\mathrm{wt}(\boldsymbol{x})$ the Hamming weight of vector $\boldsymbol{x}$ over E. For a (binary) code C denote by $\langle C \rangle$ the linear envelope of words of C over F_2. The dimension of space $\langle C \rangle$ is called the *rank* of C over F_2 and is denoted $\mathrm{rank}(C)$.

Denote by (n,w,d,N) a binary constant weight code C of length n, with weight of all codewords w, with minimum distance d and cardinality N.

The binary (n, d, N)-code A which is a linear k-dimensional space over F_2 is denoted by $[n, k, d]$-code. For any (n, d, N)-code (linear, nonlinear, or constant weight) denote by $C^\perp$ its dual code:

$$C^\perp = \{\boldsymbol{v} \in F_2^n : (\boldsymbol{v} \cdot \boldsymbol{c}) = 0, \ \forall \ \boldsymbol{c} \in C\}.$$

where

$$(\boldsymbol{v} \cdot \boldsymbol{c}) = v_1 c_1 + \cdots + v_n c_n$$

Clearly $C^\perp$ is a linear $[n, n-k, d^\perp]$-code with some minimum distance $d^\perp$, where $k = \text{rank}(C)$.

Denote by E_2^n the set of all binary vectors of length n of weight 2. Let $J_n = \{1, 2, \ldots, n\}$ be the coordinate set of E^n and let $\mathcal{S}_n$ be the full group of permutations of n elements (thus $|\mathcal{S}_n| = n!$). A binary incidence matrix of a Steiner system $S(v, 4, 3)$ is an optimal constant weight $(v, 4, 4, v(v-1)(v-2)/24)$-code C.

In this note, the Steiner system $S(v, 4, 3)$ is identified with the constant weight $(v, 4, 4, v(v-1)(v-2)/24)$-code, which uniquely defines this system (Semakov-Zinoviev [1969]).

Definition 1 *Two Steiner systems (J, B) and (J', B') of order n are isomorphic, if their incidence matrices S and S' are equivalent as constant weight codes, i.e. if there exists some permutation $\tau \in \mathcal{S}_n$ such that S and $\tau S'$ coincide up to the permutation of rows.*

3. Switching constructions of $\mathbf{SQS}(v)$.

Let C be a Steiner system $S(v,4,3)$ of 2-rank $r \le v-2$. By proper permutation of coordinates, C can be presented in the form, when the $[v, v/2, 2]$-code $C^{\perp}$, orthogonal to $\langle C \rangle$, is

$$C^{\perp} \;=\; \{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_1+\boldsymbol{u}_2\}, \qquad (1)$$

where $\boldsymbol{u}_0$ is the zero vector, $\boldsymbol{u}_1 = (11\ldots1|00\ldots0)$, and $\boldsymbol{u}_2 = (00\ldots0|11\ldots1)$. Thus we split v coordinates into two blocks of $v/2$ coordinates such that any $\boldsymbol{c} \in C$ consists of two vectors $\boldsymbol{c} = (\boldsymbol{c}_1 \,|\, \boldsymbol{c}_2)$ where each vector $\boldsymbol{c}_i$ satisfies to the overall parity checking:

$$\mathrm{wt}(\boldsymbol{c}_i) \;\equiv 0 \;(mod\ 2),\;\; i=1,2$$

(we call it a *parity rule*).

Definition 2 *Define the following (constant weight) $(8,4,4,8)$-codes:*

$$C_P = \left\{ \begin{array}{cc} (1\,1\,1\,1\,|\,0\,0\,0\,0), & (0\,0\,0\,0\,|\,1\,1\,1\,1), \\ (1\,1\,0\,0\,|\,1\,1\,0\,0), & (0\,0\,1\,1\,|\,0\,0\,1\,1), \\ (1\,0\,1\,0\,|\,1\,0\,1\,0), & (0\,1\,0\,1\,|\,0\,1\,0\,1), \\ (1\,0\,0\,1\,|\,0\,1\,1\,0), & (0\,1\,1\,0\,|\,1\,0\,0\,1) \end{array} \right\},$$

and

$$C_N = \left\{ \begin{array}{cc} (1\,1\,1\,0\,|\,1\,0\,0\,0), & (1\,1\,0\,1\,|\,0\,1\,0\,0), \\ (1\,0\,1\,1\,|\,0\,0\,1\,0), & (0\,1\,1\,1\,|\,0\,0\,0\,1), \\ (1\,0\,0\,0\,|\,1\,1\,1\,0), & (0\,1\,0\,0\,|\,1\,1\,0\,1), \\ (0\,0\,1\,0\,|\,1\,0\,1\,1), & (0\,0\,0\,1\,|\,0\,1\,1\,1) \end{array} \right\}.$$

Note that the codes C_P and C_N above differ by the permutations of the columns with numbers 4 and 5, what is equivalent, by interchanging of the elements 0 and 1 in these columns (i.e. by switching of these two columns).

For a given permutation $\pi \in \mathcal{S}_4$ denote by $C_{\pi(P)}$ (respectively, by $C_{\pi(N)}$) the code obtained from C_P (respectively, from C_N) by applying π to the last 4 columns of the code C_P (respectively C_N).

Note also that the middle six columns of C_P define two Pasch configurations.

Theorem 1 (*switching construction*).
Let S be a Steiner system $S(v,4,3)$ of 2-rank $r \leq v-2$. and let C be the corresponding constant weight $(v,4,4,v(v-1)(v-2)/24)$-code with orthogonal code (1), i.e. all codewords $\boldsymbol{c} = (\boldsymbol{c}_1 \,|\, \boldsymbol{c}_2)$ from C satisfy the parity rule. Assume that C contains as a subcode the code $C_{\pi(P)}$ for some $\pi \in \mathcal{S}$. Define the new code

$$C^*(\pi(P)) \;=\; (C \setminus C_{\pi(P)}) \cup C_{\pi(N)}.$$

Then:

1). *The set $C^* = C^*(\pi(P))$ is a constant weight $(v,4,4,v(v-1)(v-2)/24)$-code, which defines a new Steiner system $S(v,4,3)$, denoted by $S^* = S^*(\pi(P))$.*

2). *The new system S^* is not isomorphic to the initial system S (since they have different number of Pasch configurations).*

3). *If the initial system S is resolvable and if the code $C_{\pi(P)}$ belongs to exactly four parallel classes of C, then the resulting system S^* is resolvable too.*

4). *Let the initial system S has a 2-rank $r = v - 2$ and let it is divided into two parts, such that each part satisfies the parity rule. If the first four nonzero positions of $C_{\pi(P)}$ belong to the left hand side of C and the rest four nonzero positions of $C_{\pi(P)}$ belong to the right hand side of C, then the 2-rank r^* of resulting system S^* is increasing, i.e. $r^* = r + 1 = v - 1$.*

4. The structure of Steiner systems $S(v,4,3)$ with rank $v-1$ over F_2.

Let $S = S(v,4,3)$ be of rank $v-1$ over F_2. Divide the coordinate set $J = \{1,2,\ldots,v\}$ of S into two arbitrary equal halves: J_1 and J_2. Applying some permutation $\pi \in \mathcal{S}_n$, any vector $\boldsymbol{c} \in \pi(C)$ can be presented in the form $\boldsymbol{c} = (\boldsymbol{c}_1 \,|\, \boldsymbol{c}_2)$, where $\mathrm{supp}(\boldsymbol{c}_i) \in J_i$ for $i = 1,2$. Hence without loss of generality assume that J_1 is the left half of J and J_2 is the right half of J_2.

Definition 3 *For any Steiner system $S(v,4,3)$ of rank $v-1$ over F_2 define the left and right spectrum (x_i, y_i, z_i), $i = 1,2$ as follows:*

$$
\begin{aligned}
x_i &= \left|\{\boldsymbol{c} = (\boldsymbol{c}_1 \,|\, \boldsymbol{c}_2) : \ \mathrm{wt}(\boldsymbol{c}_i) = 4\}\right|, \\
y_i &= \left|\{\boldsymbol{c} = (\boldsymbol{c}_1 \,|\, \boldsymbol{c}_2) : \ \mathrm{wt}(\boldsymbol{c}_i) = 3\}\right|, \\
z_i &= \left|\{\boldsymbol{c} = (\boldsymbol{c}_1 \,|\, \boldsymbol{c}_2) : \ \mathrm{wt}(\boldsymbol{c}_i) = 2\}\right|.
\end{aligned}
$$

Lemma 1 *Let S be an arbitrary Steiner system $S(v,4,3)$ of 2-rank $v-1$ over F_2. Then $x = x_1 = x_2$, $y = y_1 = y_2$, $z = z_1 = z_2$. Furthermore*

$$y = \binom{v/2}{3} - 4\,x, \quad z = 6\,x + \binom{v/2}{2}. \tag{2}$$

Clearly for the same system the numbers x, y and z depend on the choice of subsets J_i.

Definition 4 *We say that* 4 *different binary vectors of length* v *and weight* 3 *form a* 4*-clique, if*

$$|\bigcup_{i=1}^{4} \mathrm{supp}(\boldsymbol{y}_i)| \;=\; 4.$$

Lemma 2 *Let* X *be a constant weight* $(v,4,4,x)$ *code with cardinality*

$$x \le v(v-1)(v-2)/24 - 2.$$

Denote by Y *the constant weight* $(v,3,2,y)$ *code, formed by all vectors of weight* 3*, which are not covered by codewords of* X*, i.e.*

$$y = \binom{v}{3} - 4\,x.$$

Then X *can be imbedded into a Steiner system* $S(v,4,3)$*, if and only if all the codewords of* Y *can be partitioned into disjoint* 4*-cliques* $C_1,\ldots,C_k,\;\; k = y/4$*, such that*

$$|\mathrm{supp}(C_i) \cap \mathrm{supp}(C_j)| \le 2 \quad \textit{for any } i \ne j.$$

5. Induced Steiner systems $S(v, 4, 3)$.

We say that a Steiner system $S = S(v, 4, 3)$ of full rank $r = v - 1$ is *induced*, if it is obtained by the switching construction from some Steiner system $S' = S(v, 4, 3)$ of rank $\leq v - 2$. In the contrary case, we call this system *singular*.

Theorem 2 *Let $S = S(v, 4, 3)$ be a Steiner system of rank $r = v - 1$ over F_2 with spectrum (x, y, z) and let v is a multiple of 4. Let X_i and Y_j be the corresponding $(v/2, 4, 4, x)$- and $(v/2, 3, 2, y)$-codes, where y satisfies (2) and $i, j \in \{1, 2\}$. If X_1 and X_2 are any subcodes of a Steiner system $S' = S(v/2, 4, 3)$, then S is an induced system.*

It is known (Z-Z [2006], Kaski-Östergård-Pottonen [2006]) that there are exactly 708103 non-isomorphic Steiner systems SQS(16) of rank 14 over F_2. By computations it was found that all these 708103 systems give 295488 different double Pasch configurations. For each system SQS(16) of rank 14, containing some double Pasch configurations we have applied all possible switchings.

Theorem 3 (*Computational results*). *There are* 305616 *non-isomorphic induced Steiner systems* $S(16, 4, 3)$ *of rank* 15 *over* F_2. *They are obtained from* 708103 *non-isomorphic Steiner systems* $SQS(16)$ *of rank* 14 *over* F_2 *by applying all possible switchings.*

Remark 1 *Taking into account the result of (Kaski-Östergård-Pottonen [2006]) we conclude that there are exactly* 27715 *non-isomorphic singular Steiner systems* $S(16, 4, 3)$ *of rank* 15.

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